Derivative interactions of massive gravity

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IAS, HKUST

[arXiv:1403.6781] and work in progress

GR is special

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Linearized GR

$$\mathcal{L}_{\mathrm{linear}} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h$$
$$+\frac{1}{2}m^2\left(h^2 - h_{\mu\nu}h^{\mu\nu}\right)$$

Fierz-Pauli mass term

Problems and difficulties (1)

- **van Dam-Veltman-Zakharov (vDVZ) discontinuity**
  The helicity-0 mode couples to the trace of the matter energy-momentum tensor with the same strength as the helicity-2 modes, which prevents the theory from recovering linearized GR in the massless limit.

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A tension between **nonlinearity** and the **health** of the theory.

Problems and difficulties (2)

Degrees of freedom in GR:

$$\mathbf{10 - 4 - 4 = 2}$$

4 gauge d.o.f

4 constraints

2 polarizations of a massless graviton

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The notion of mass for gravity requires a reference background $\bar{g}_{\mu\nu}$ **.**
Any given background breaks General Covariance.

$$\mathcal{L} = \frac{1}{2} R\left[g_{\mu\nu}\right] + U\left(g_{\mu\nu}, \bar{g}_{\mu\nu}\right)$$

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Degrees of freedom for a general mass term:

10 - 4 = 5 + 1 → **Boulware-Deser (BD) ghost**

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Hamiltonian of GR

$$\mathcal{H}_{\rm GR} \simeq N\left[\frac{2}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i_i\pi^j_j\right) - \frac{1}{2}\sqrt{h}R^{(3)}\left[h\right]\right] - \sqrt{h}N_i\nabla_j\left(\frac{2}{\sqrt{h}}\pi^{ij}\right)$$

For a general mass term $U\left(N, N_i, h_{ij}\right)$, {$N$, N_i} are NOT Lagrange multiplier any more.
All components of $h_{\mu\nu}$ get excited.

dGRT: Stueckelberg analysis

Ghost-free nonliear mass terms (dRGT)

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Metric perturbation (around Minkowski): $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$

Covariantized metric perturbation

$$h_{\mu\nu} \to H_{\mu\nu} \equiv \eta_{ab} \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^b}{\partial x^\nu}$$

$$\phi^a = \delta^a_\mu \left(x^\mu - \pi^\mu\right)$$

Goldstones

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Covariantized mass terms

$$U\left(\eta_{\mu\nu}, h_{\mu\nu}\right) \to U\left(g_{\mu\nu}, H_{\mu\nu}\right)$$

Covariantization of Fierz-Pauli term:

$$[h]^2 - [h^2] \to \sqrt{-g}\left([H]^2 - [H^2]\right) \qquad [h] = h^\mu_\mu, \qquad [h^2] = h_{\mu\nu} h^{\mu\nu}$$

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Necessary condition for ghost-free in the Stueckelberg analysis:
In the decoupling limit, the helicity-0 mode π (defined by $\pi_\mu = \partial_\mu \pi$) has NO high order equation of motion.
→ π has no self-interactions!

Stueckelberg expansions

2nd order:

$$\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi$$

$$\frac{\sqrt{-g}}{4}\left([H]^2 - [H^2]\right)$$

$$\xrightarrow{\text{decoupling limit}} \left([\Pi]^2 - [\Pi^2]\right) + \left([\Pi^3] - [\Pi][\Pi^2]\right) + \frac{1}{4}\left([\Pi^2]^2 - [\Pi^4]\right) + \mathcal{O}\left(\pi^5\right)$$

total derivative *ghost reappears at nonlinear order*

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Adding higher order compensation terms.

$$\frac{\sqrt{-g}}{8}\left([H]\,[H^2] - [H^3]\right)$$

$$\xrightarrow{\text{decoupling limit}} [\Pi]\,[\Pi^2] - [\Pi^3] + \left(\frac{3}{2}[\Pi^4] - [\Pi]\,[\Pi^3] - \frac{1}{2}[\Pi^2]^2\right) + \mathcal{O}\left(\pi^5\right)$$

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A ghost-free combination up to the 3rd order

$$\sqrt{-g}\left[\frac{1}{4}\left([H]^2 - [H^2]\right) + \frac{1}{8}\left([H][H^2] - [H^3]\right)\right]$$

$$\xrightarrow{\text{decoupling limit}} \left(\frac{5}{4}[\Pi^4] - \frac{1}{4}[\Pi^2]^2 - [\Pi][\Pi^3]\right) + \mathcal{O}\left(\pi^5\right).$$

ghost at 4th order

dRGT as resummation

This procedure can be performed at arbitrarily high order.

$$\frac{1}{4}\left([H]^2-[H^2]\right)$$

→ **Covariantization** of Fierz-Pauli term

$$\begin{aligned}
&+\frac{1}{8}\left([H]\left[H^2\right]-\left[H^3\right]\right)\\
&+\frac{1}{64}\left(\left[H^2\right]^2+4[H]\left[H^3\right]-5\left[H^4\right]\right)\\
&+\frac{1}{128}\left(5[H]\left[H^4\right]+2\left[H^2\right]\left[H^3\right]-7\left[H^5\right]\right)\\
&+\frac{1}{512}\left(14[H]\left[H^5\right]+5\left[H^2\right]\left[H^4\right]+2\left[H^3\right]^2-21\left[H^6\right]\right)\\
&+\mathcal{O}\left(H^7\right)
\end{aligned}$$

$$\rightarrow\ [\mathcal{K}]^2-\left[\mathcal{K}^2\right]$$

→ **Nonlinear completion** of Fierz-Pauli term

$$\mathcal{K}^\mu_\nu \equiv \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} \equiv \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2\,(2n-1)}\left(H^n\right)^\mu_\nu$$

The full dRGT terms

Full Lagrangian for the nonlinear ghost-free massive gravity

$$\mathcal{L} = \frac{1}{2}R + m^2\left(\mathcal{L}^{\mathrm{dRGT},2} + \alpha_3\mathcal{L}^{\mathrm{dRGT},3} + \alpha_4\mathcal{L}^{\mathrm{dRGT},4}\right)$$

$$\begin{aligned}
\mathcal{L}^{\mathrm{dRGT},2} &= \mathcal{K}^2 - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}, \\
\mathcal{L}^{\mathrm{dRGT},3} &= \mathcal{K}^3 - 3\mathcal{K}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + 2\mathcal{K}^{\mu}_{\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{K}^{\rho}_{\mu}, \\
\mathcal{L}^{\mathrm{dRGT},4} &= \mathcal{K}^4 - 6\mathcal{K}^2\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + 3\left(\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}\right)^2 + 8\mathcal{K}\mathcal{K}^{\mu}_{\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{K}^{\rho}_{\mu} - 6\mathcal{K}^{\mu}_{\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{K}^{\rho}_{\sigma}\mathcal{K}^{\sigma}_{\mu}.
\end{aligned}$$

Hamiltonian analysis (1)

The most general mass terms at the 2nd and 3rd orders:

$$\begin{aligned}\mathcal{L}_2^{\rm pot} &= b_1 h^\mu_\mu h^\nu_\nu + b_2 h_{\mu\nu} h^{\mu\nu}, \\ \mathcal{L}_3^{\rm pot} &= c_1 h^\mu_\mu h^\nu_\nu h^\rho_\rho + c_2 h^\rho_\rho h^\mu_\nu h^\nu_\mu + c_3 h^\mu_\nu h^\nu_\rho h^\rho_\mu,\end{aligned}$$

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Using ADM variables:

$$ds^2 = -\left(N^2 - \gamma^{ij} N_i N_j\right) dt^2 + 2 N_i dt dx^i + \gamma_{ij} dx^i dx^j$$

$$N \equiv 1 + \alpha, \qquad N_i \equiv \beta_i, \qquad \gamma_{ij} \equiv \delta_{ij} + h_{ij}$$

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In GR, it is the lapse *N=1+*α that acts as a Lagrange multiplier in Hamiltonian.
The mass terms should be tuned to be compatible with this fact.

$$\begin{aligned}\mathcal{L}^{\rm pot,ADM} = \; & 4(b_1 + b_2)\,\alpha^2 + 4b_1 \alpha h_{ii} + 2(b_1 + 6c_1 + 2c_2)\,\alpha^2 h_{ii} \\ & -2b_2 \beta_i \beta_i - 2\,(2c_2 + 3c_3)\,\alpha \beta_i \beta_i + b_1 h_{ii} h_{jj} + b_2 h_{ij} h_{ij} \\ & +8(c_1 + c_2 + c_3)\,\alpha^3 \\ & +2\alpha\,(3c_1 h_{ii} h_{jj} + c_2 h_{ij} h_{ij}) - 3c_3 h_{ij} \beta_i \beta_j - 2\,(b_1 + c_2)\, h_{ii} \beta_j \beta_j \\ & +c_1 h_{ii} h_{jj} h_{kk} + c_2 h_{ii} h_{jk} h_{jk} + c_3 h_{ij} h_{jk} h_{ki} \\ & +\mathcal{O}(4)\end{aligned}$$

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Requiring α appears linearly in the Hamiltonian yields 3 constraints:

$$\begin{aligned} b_1 + b_2 &= 0, \\ b_1 + 6c_1 + 2c_2 &= 0, \\ c_1 + c_2 + c_3 &= 0. \end{aligned}$$

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"Potentially" Ghost-free mass terms up to the 3rd order:

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\mathcal{L}^{\text{pot}} = b_1 &\left[\left(h^\mu_\mu h^\nu_\nu - h_{\mu\nu} h^{\mu\nu} \right) + \frac{1}{2} \left(h^3 - 4h\, h_{\mu\nu} h^{\mu\nu} + 3 h^\mu_\nu h^\nu_\rho h^\rho_\mu \right) \right] \\
&+ \left(c_1 - \frac{1}{2} b_1 \right) \left(h^3 - 3h\, h_{\mu\nu} h^{\mu\nu} + 2 h^\mu_\nu h^\nu_\rho h^\rho_\mu \right) + \mathcal{O}\left(h^4\right).
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Hamiltonian analysis (3)

Nonlinear completion

expansion of $\mathcal{K}^2 - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}$

expansion of $\mathcal{K}^3 - 3\mathcal{K}\mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + 2\mathcal{K}^{\mu}_{\nu}\mathcal{K}^{\nu}_{\rho}\mathcal{K}^{\rho}_{\mu}$

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Pseudo-linear counterpart of FP at 3rd order

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"Pseudo-linear" mass terms

Nonlinear in $h_{\mu\nu}$, but respecting the linearized diffeomorphism:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

K. Hinterbichler, JHEP 1310 (2013) 102,

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- We will have a more general class of massive gravity theories beyond the dRGT one, as well as derivative interactions for multiple metrics.

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\sqrt{-g}R\left[g\right] + \frac{\tilde{M}_{\rm pl}^2}{2}\sqrt{-f}R\left[f\right] + U\left(g^{\mu\rho}f_{\rho\nu}\right)$$

potential interactions

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A missing block?

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Perturbative expansion

Linearized GR

~ ∂h∂h

A missing block?

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Fierz-Pauli
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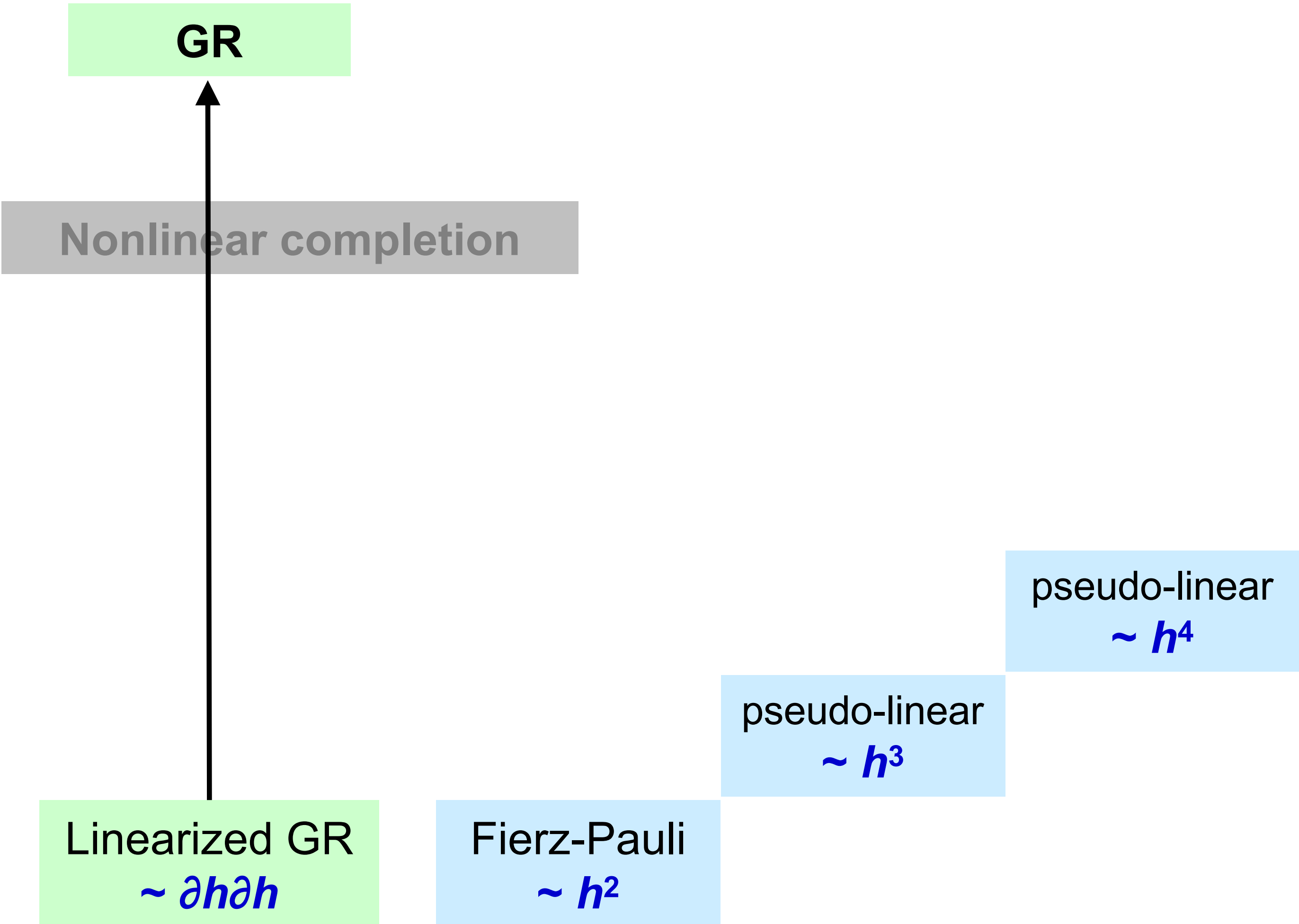
A missing block?



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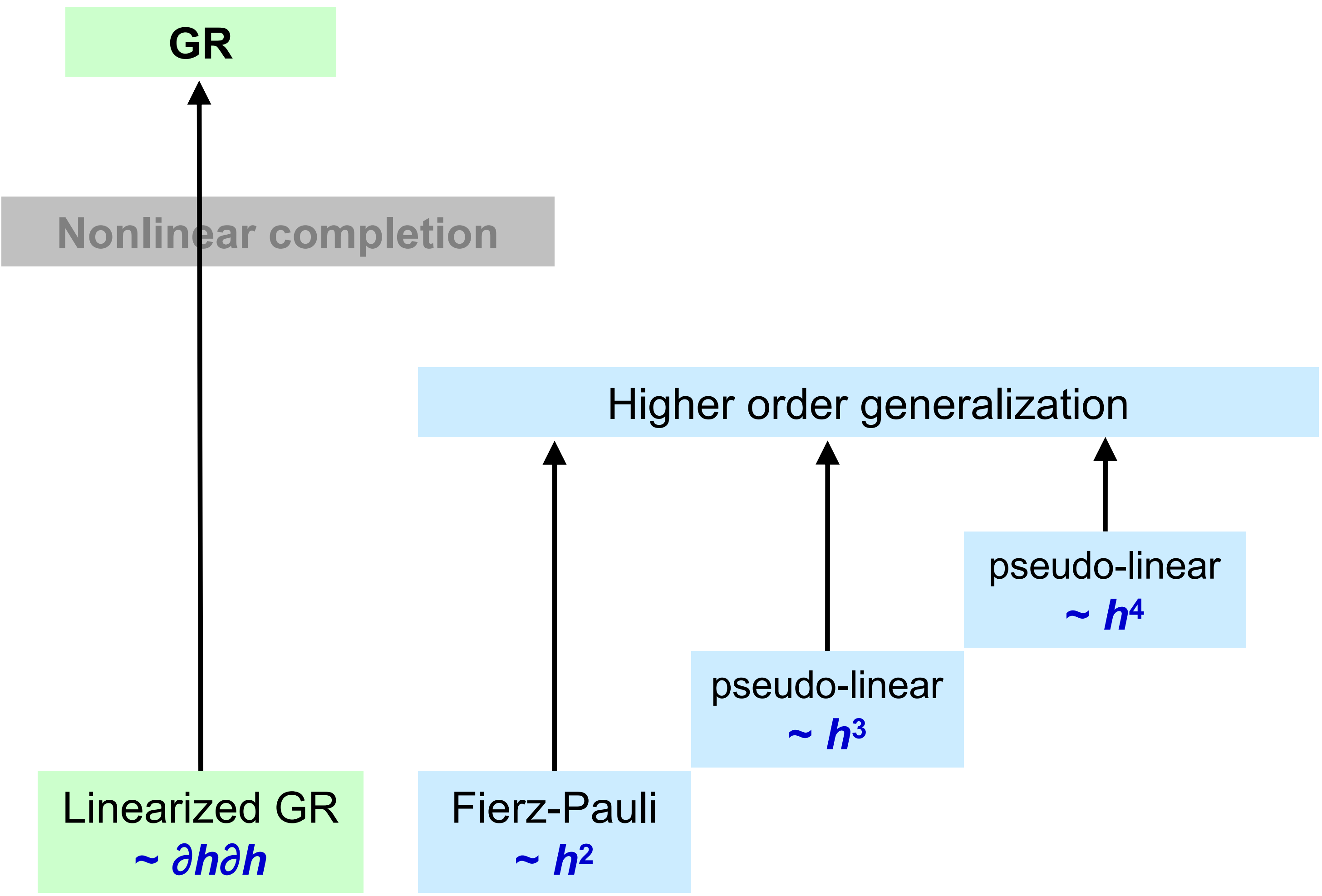

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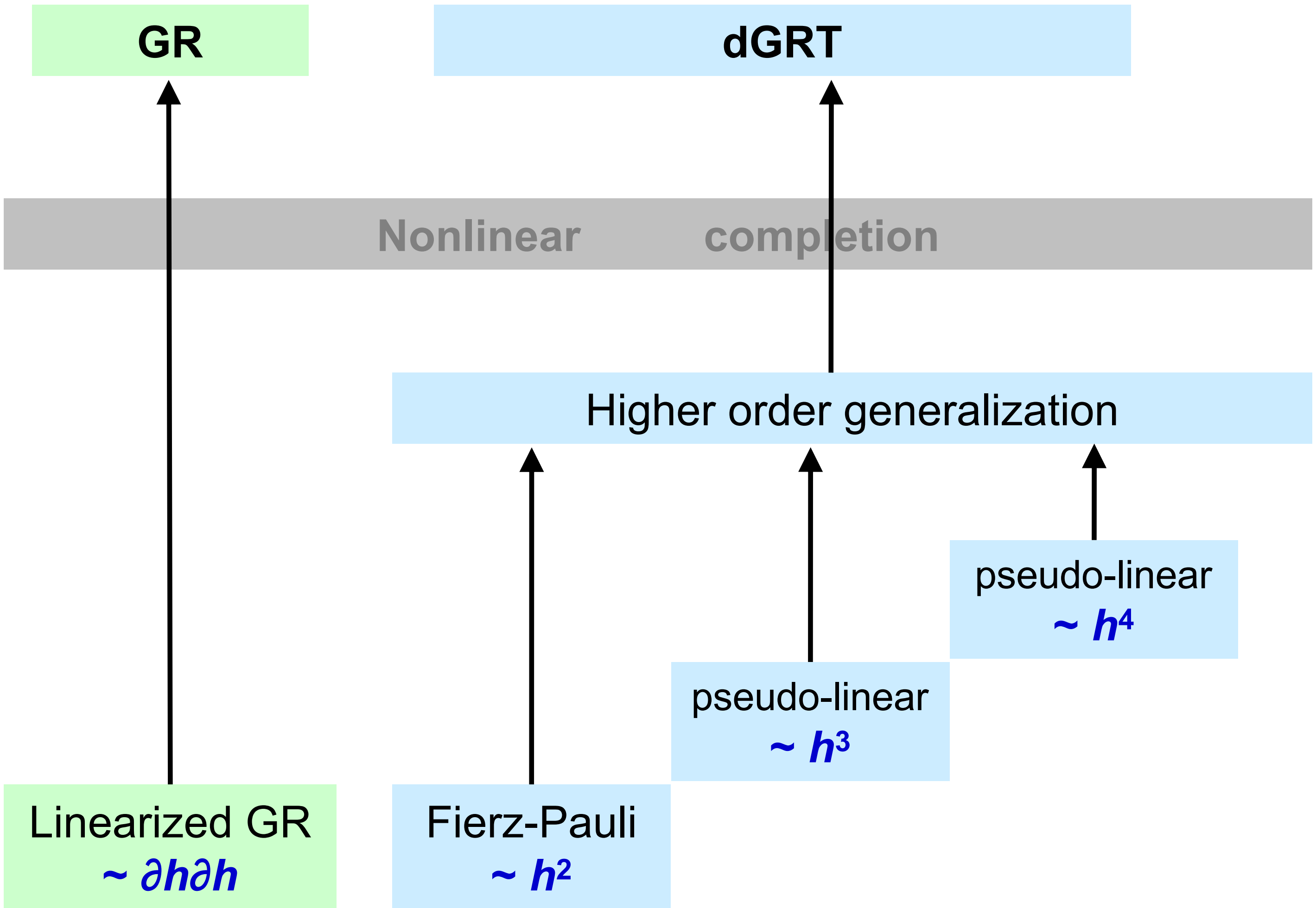

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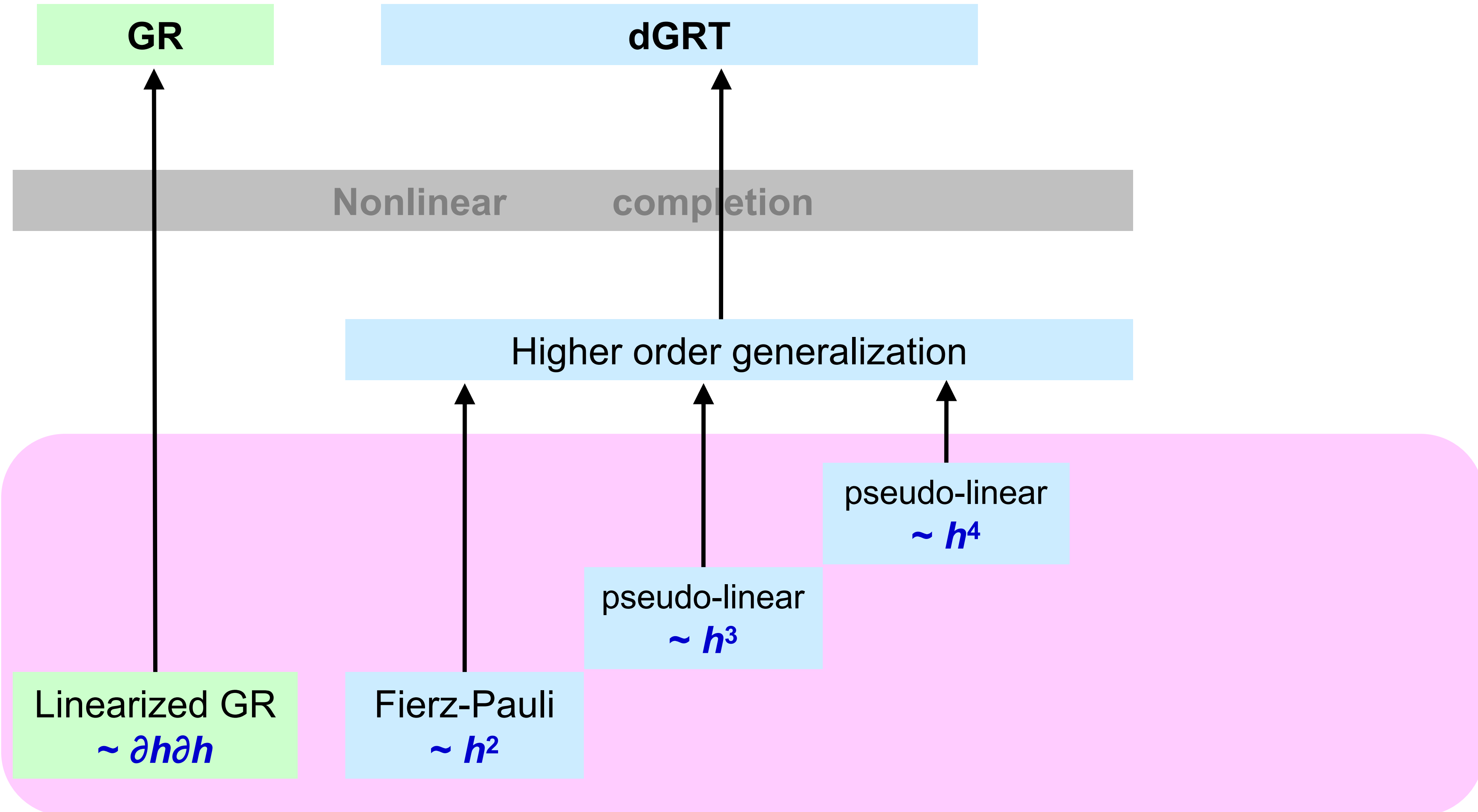
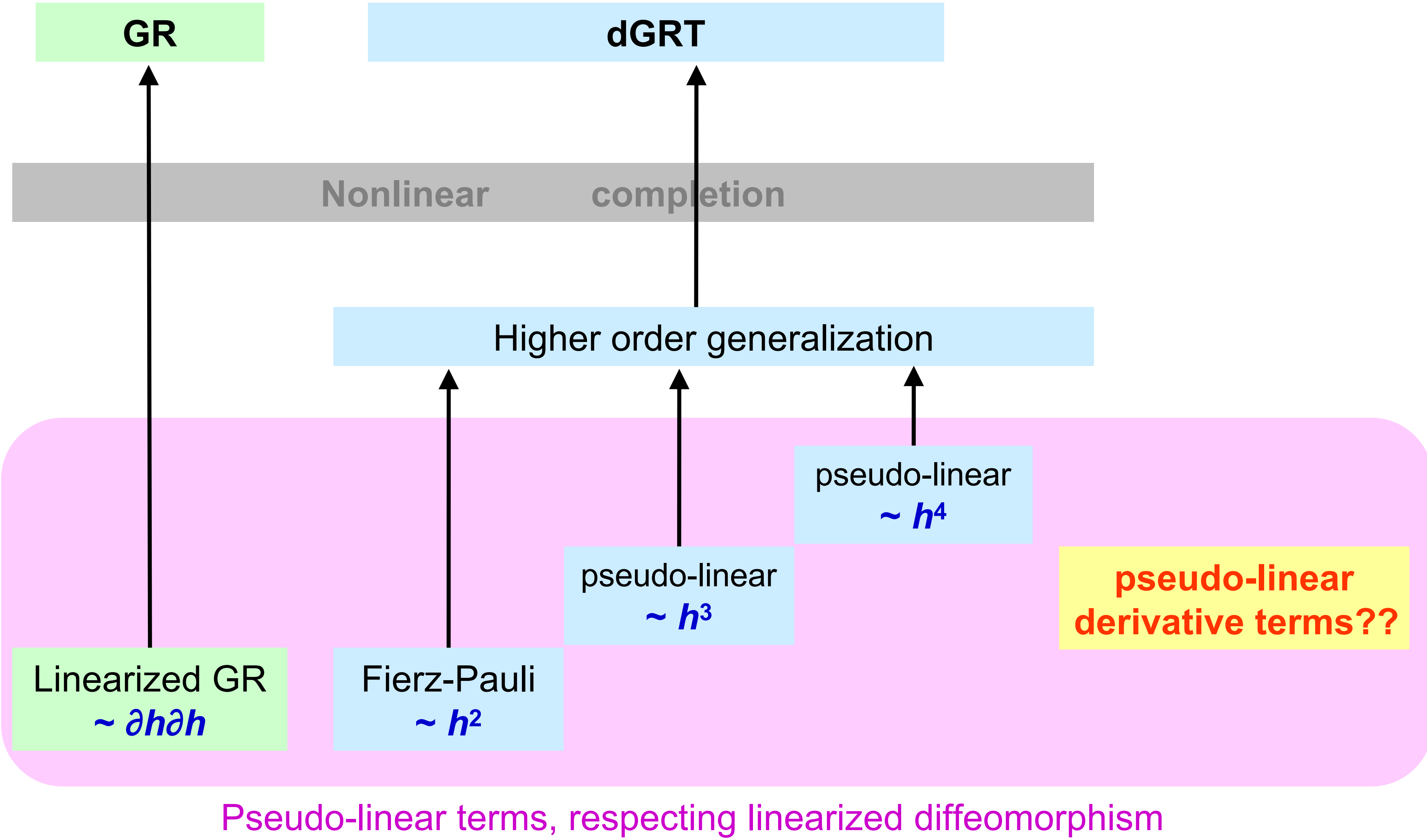


Pseudo-linear terms, respecting linearized diffeomorphism

A missing block?



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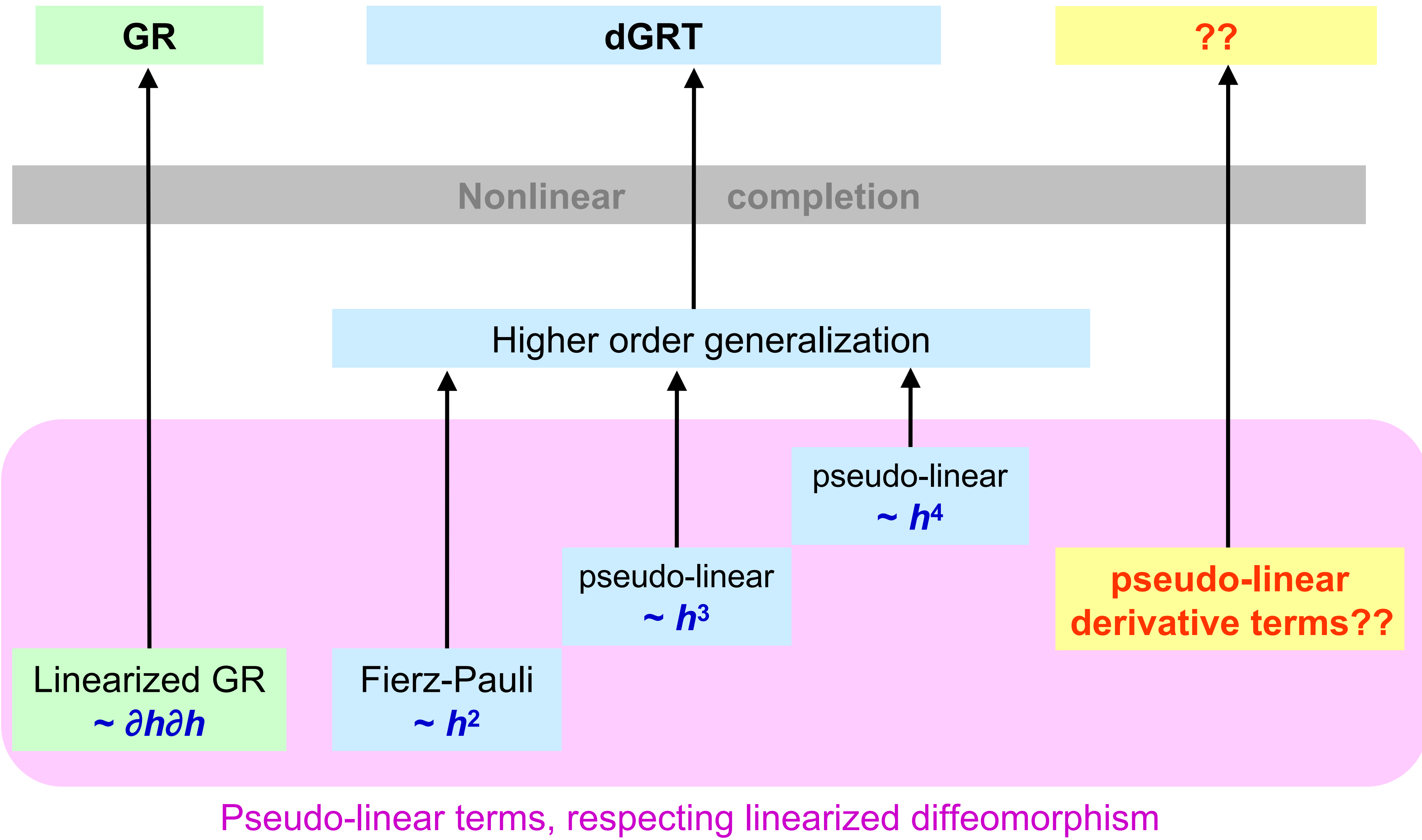

Ansatz at cubic order

The most general Lorentz-invariant cubic derivative terms, with 14 parameters:

$$
\begin{aligned}
\mathcal{L}_3^{\mathrm{der}} \;=\; & c_1 h^{\alpha\beta}\partial_\alpha h^{\lambda\mu}\partial_\beta h_{\lambda\mu} + c_2 h^{\alpha\beta}\partial_\alpha h^\lambda_\lambda \partial_\beta h^\mu_\mu + c_3 h^{\alpha\beta}\partial_\beta h^\lambda_\alpha \partial_\lambda h^\mu_\mu + c_4 h^{\alpha\beta}\partial_\lambda h^\mu_\mu \partial^\lambda h_{\alpha\beta} \\
& + c_5 h^\alpha_\alpha \partial_\lambda h^\mu_\mu \partial^\lambda h^\beta_\beta + c_6 h^{\alpha\beta}\partial_\lambda h^\lambda_\alpha \partial_\mu h^\mu_\beta + c_7 h^{\alpha\beta}\partial_\beta h^\lambda_\alpha \partial_\mu h^\mu_\lambda + c_8 h^\alpha_\alpha \partial_\beta h^{\beta\lambda}\partial_\mu h^\mu_\lambda \\
& + c_9 h^{\alpha\beta}\partial^\lambda h_{\alpha\beta}\partial_\mu h^\mu_\lambda + c_{10} h^\alpha_\alpha \partial^\lambda h^\beta_\beta \partial_\mu h^\mu_\lambda + c_{11} h^{\alpha\beta}\partial_\lambda h_{\beta\mu}\partial^\mu h^\lambda_\alpha + c_{12} h^{\alpha\beta}\partial_\mu h_{\beta\lambda}\partial^\mu h^\lambda_\alpha \\
& + c_{13} h^\alpha_\alpha \partial_\lambda h_{\beta\mu}\partial^\mu h^{\beta\lambda} + c_{14} h^\alpha_\alpha \partial_\mu h_{\beta\lambda}\partial^\mu h^{\beta\lambda}.
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Requiring the cubic Lagrangian

$$\mathcal{L}_3^{\rm der,ADM} = \mathcal{L}_3^{\rm der,ADM}(\alpha, \beta_i, h_{ij})$$

contains no time derivatives of α and β_i yields 12 constraints.

$$\begin{aligned} c_2 &= -c_1, \quad c_3 = 4c_1, \quad c_4 = -2c_1, \quad c_6 = -5c_1 + 2c_5,\\ c_7 &= -4c_1, \quad c_8 = -3c_1 + 2c_5, \quad c_9 = 2c_1, \quad c_{10} = -2c_5,\\ c_{11} &= 3c_1 - 2c_5, \quad c_{12} = 2c_1, \quad c_{13} = 3c_1, \quad c_{14} = -c_5.\end{aligned}$$

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Requiring the cubic Lagrangian

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\end{aligned}
$$

With this choice of parameters, there is a constraint in the Hamiltonian:

$$
\det\left(\frac{\partial^2 \mathcal{H}^{\mathrm{ADM}}}{\partial n_a \partial n_b}\right) = 0, \qquad \text{with} \qquad n_a \equiv \{\alpha, \beta_i\}
$$

which may eliminate the BD ghost.

Derivative terms at cubic order

"Potentially" ghost-free derivative terms up to the 3rd order:

$$\mathcal{L}_2^{\mathrm{der}} + \mathcal{L}_3^{\mathrm{der}} = -\frac{b_1^3}{\left(c_5 - c_1\right)^2}\left(\mathcal{L}_2^{\mathrm{GR}}[\tilde{h}] + \mathcal{L}_3^{\mathrm{GR}}[\tilde{h}] + \frac{c_1 - 2c_5}{8\left(c_5 - c_1\right)}\mathcal{L}^{\mathrm{PL}}[\tilde{h}]\right)$$

$$h_{\mu\nu} = \frac{b_1}{2\left(c_5 - c_1\right)}\tilde{h}_{\mu\nu}$$

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- $\mathcal{L}_{2,3}^{\rm GR}$ come from the expansion of the standard GR $\sqrt{-g}R$
- $\mathcal{L}^{\rm PL}[\tilde{h}]$ is the "pseudo-linear derivative terms" at the 3rd order:

$$\boxed{\mathcal{L}^{\rm PL} \simeq \eta^{\mu_1}_{[\nu_1}\eta^{\mu_2}_{\nu_2}\eta^{\mu_3}_{\nu_3}\eta^{\mu_4}_{\nu_4]} h^{\nu_1}_{\mu_1}\partial^{\nu_2}h^{\nu_3}_{\mu_3}\partial_{\mu_2}h^{\nu_4}_{\mu_4}}$$

At the 3rd order in $h_{\mu\nu}$, we indeed find non-GR derivative terms.

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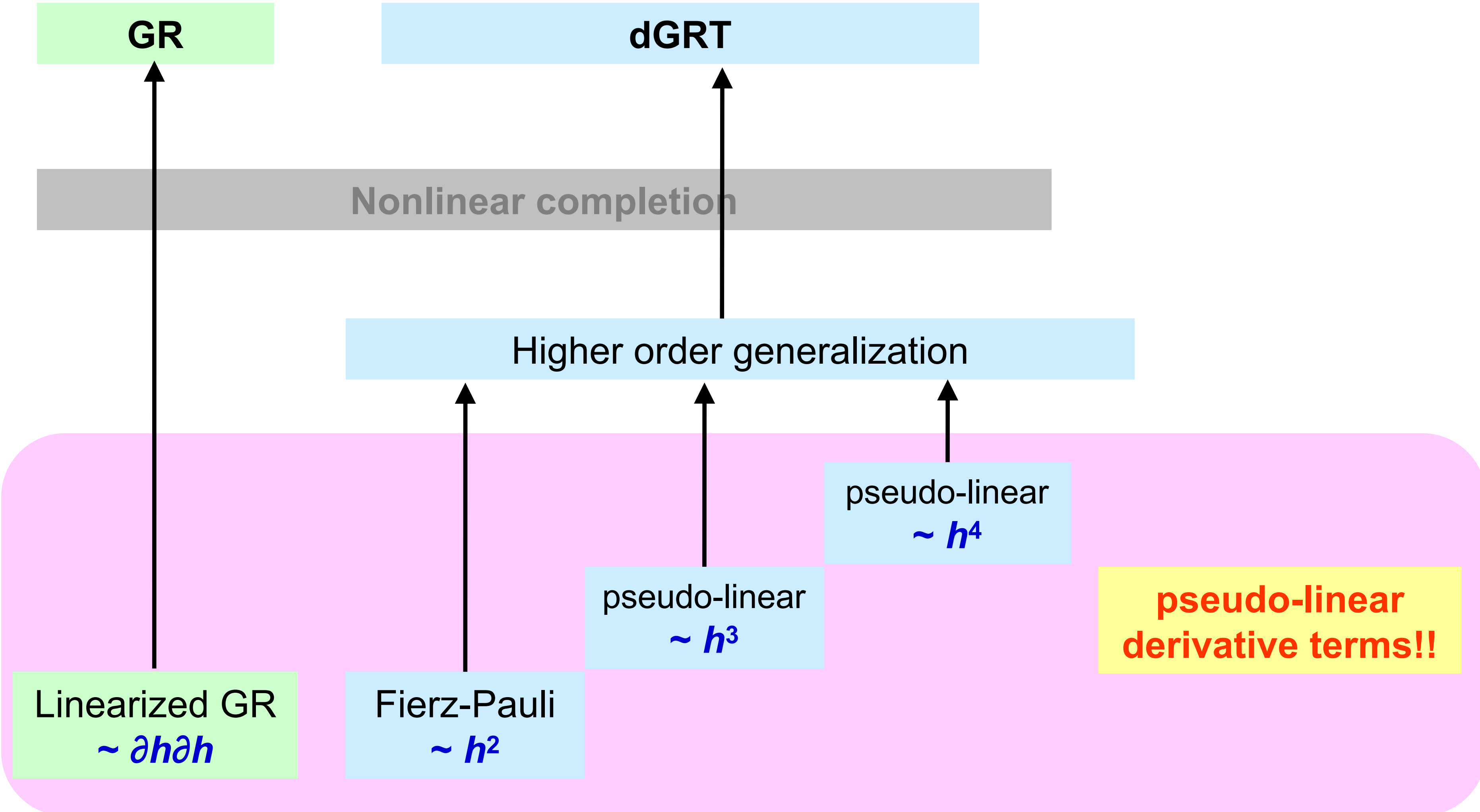
At the 3rd order in $h_{\mu\nu}$, we indeed find non-GR derivative terms.

The mass terms compatible with the derivative terms:

$$\mathcal{L}_2^{\rm pot} + \mathcal{L}_3^{\rm pot} = \frac{b_1' b_1^2}{(c_5 - c_1)^2}\left[\mathcal{L}_2^{\rm dRGT,1}[\tilde{h}] + \mathcal{L}_3^{\rm dRGT,1}[\tilde{h}] + \left(1 + \frac{b_1 c_1'}{b_1'\,(c_5 - c_1)}\right)\mathcal{L}_3^{\rm dRGT,2}[\tilde{h}]\right]$$

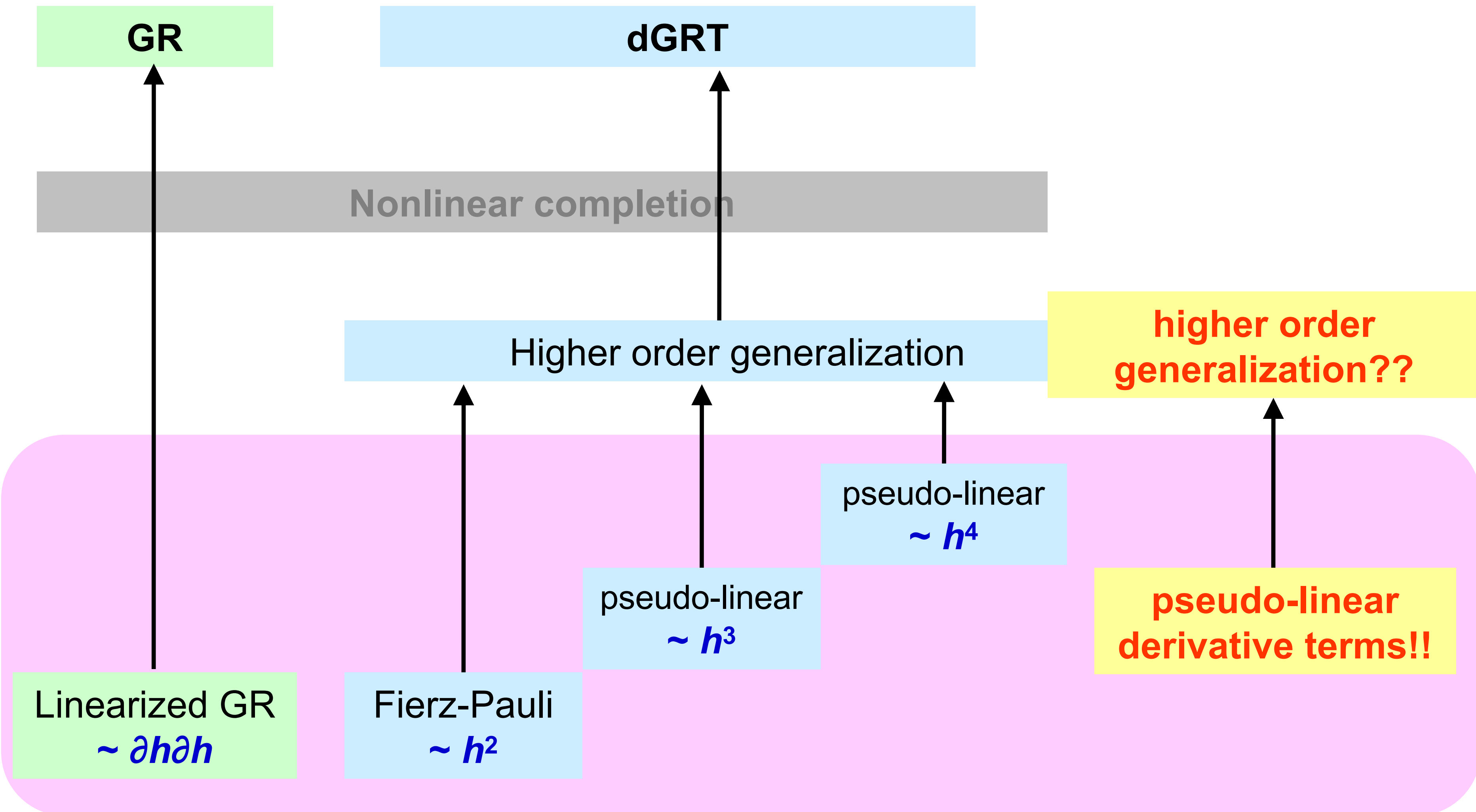
which are just the dRGT mass terms.

Nonlinear completion?



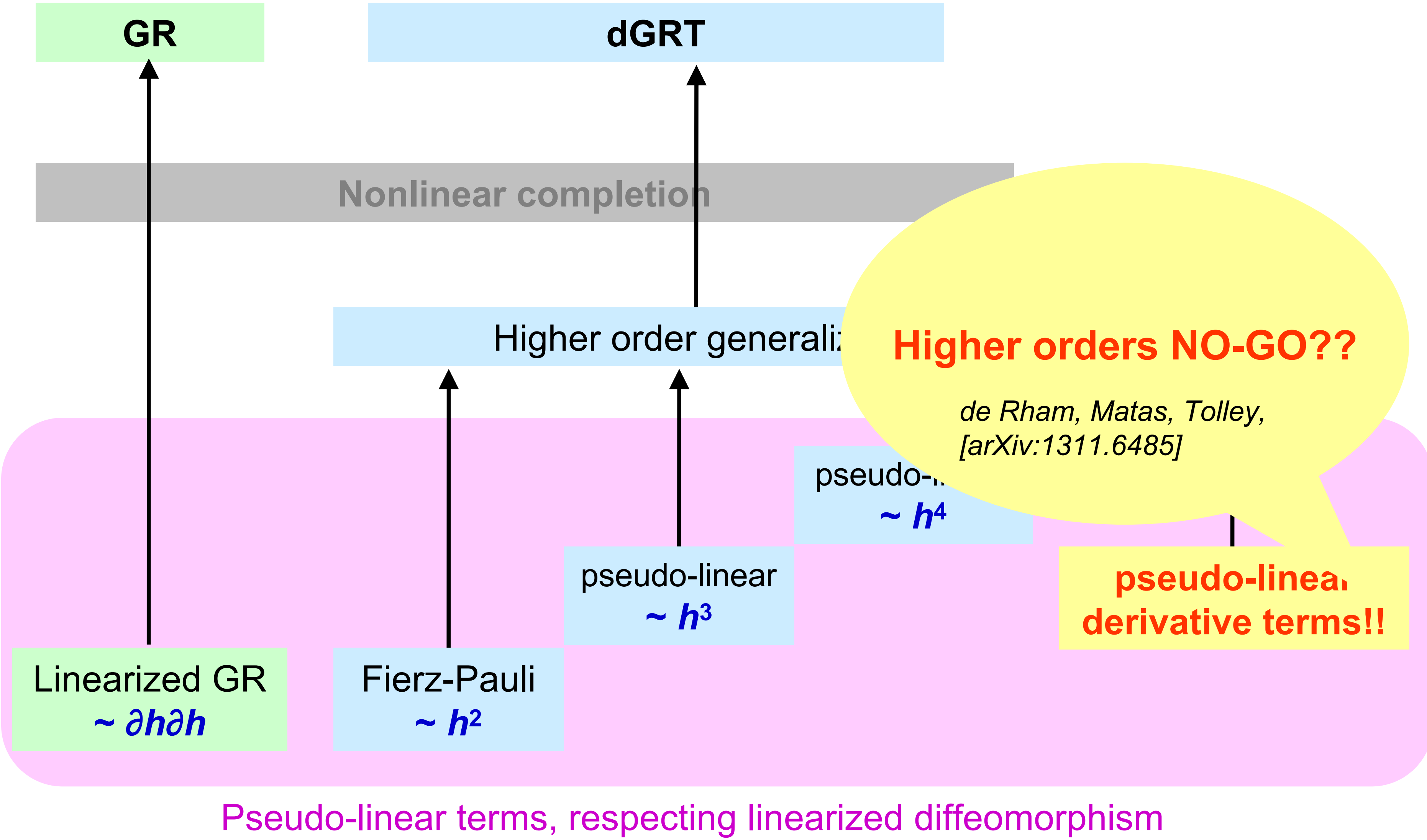
Pseudo-linear terms, respecting linearized diffeomorphism

Nonlinear completion?



Pseudo-linear terms, respecting linearized diffeomorphism

Nonlinear completion?



Perspective

- GR is the unique theory for a spin-2 field, no matter it is massless or massive.

- Pseudo-linear derivative terms are special?

- Non-flat background?

Higher orders NO-GO??

pseudo-linear derivative terms!!

Thank you for your attention!