Robust Bayes Inference for Non-Identified SVARs*

Raffaella Giacomini† and Toru Kitagawa‡

This draft: March, 2014

Abstract

This paper considers a robust Bayes inference for structural vector autoregressions, where impulse responses of interest are non-identified. The non-identified impulse responses arise if the insufficient number of equality restrictions and/or a set of sign restrictions on impulse responses are the only credible assumptions available. A posterior distribution for the set-identified impulse responses obtained via the standard Bayesian procedure remains to be sensitive to a choice of prior, even asymptotically. In order to make posterior inference free from such sensitivity concern, this paper introduces a class of priors (ambiguous belief) for the non-identified aspects of the model, and proposes to report the range of the posterior mean and posterior probability for the impulse responses as a prior varies over the class. We argue that this posterior bound analysis is a useful tool to separate the information for the impulse responses provided by the likelihood from the information provided by the prior input that cannot be updated by data. The posterior bounds we construct asymptotically converge to the true identified set, which frequentist inference in set-identified models typically concerns. In terms of implementation, the posterior bound analysis does not involve an inversion of hypothesis test, and it is therefore computationally less demanding than the frequentist confidence intervals of Moon, Schorfheide, and Granziela (2013) especially when the number of variables in the VAR is large.

Keywords: Partial Identification, Structural VAR, Robust Bayes Analysis, Lower and Upper Probabilities.

---

*We would like to thank Eleonora Granziela, Frank Kleibergen, Sophocles Mavroeidis, Frank Schorfheide, and seminar participants at the Bank of Canada, 2013 Cowles Summer Econometrics Conference, 2014 EABCN Conference, Hitotsubashi University, and Tinbergen Institute for their valuable comments. The financial supports from the ESRC through the ESRC Centre for Microdata Methods and Practice (CeMMAP) (grant number RES-589-28-0001) are gratefully acknowledged.

†University College London, Department of Economics. Email: r.giacomini@ucl.ac.uk

‡University College London, Department of Economics. Email: t.kitagawa@ucl.ac.uk

1 Introduction

The structural vector autoregressions (SVARs) proposed by Sims (1980) offer a useful tool to infer dynamic causal impacts among economic variables, and they have been widely used in macroeconomic policy analysis. A common practice of SVAR analysis assumes a sufficient number of identifying restrictions, in order to guarantee that knowledge on the sampling distribution of data can uniquely pin down the underlying structural parameters. The estimation results and the policy implications crucially rely on these identifying restrictions. Therefore, credibility of some of these assumptions often becomes a source of controversies, and, in many contexts, a uniform consensus on what set of identifying restrictions should be imposed is not available in the literature.

The main goal of this paper is to propose an inference procedure for the impulse responses, when the imposed restrictions fail to identify the underlying structural parameters. To further illustrate the motivation of this paper, consider a simple one-lag structural VAR model with three variables, $(\Delta \log P_{c,t}, \Delta \log Y_t, R_t)$, as studied by Leeper, Sims, and Zha (1996):

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \Delta \log P_{c,t} \\ \Delta \log Y_t \\ R_t \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} \Delta \log P_{c,t-1} \\ \Delta \log Y_{t-1} \\ R_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{M,t} \\ \epsilon_{Y,t} \\ \epsilon_{P,t} \end{pmatrix},$$

where $P_{c,t}$ is the price index of commodities, Y_t output, and R_t the nominal short-term interest rate. The first equation is the monetary policy equation, where structural shock $\epsilon_{M,t}$ stands for a contractionary monetary policy shock. The second equation is the output equation characterizing the behavior of the final good producers, and $\epsilon_{Y,t}$ is a structural productivity shock. The third equation is the price equation implied from an equilibrium condition, and $\epsilon_{P,t}$ is a price shock. Following a necessary and sufficient condition for exact-identification as demonstrated in Rubio-Ramirez, Waggoner, and Zha (2010), the following set of restrictions yields identification of the structural parameters,

(A1) $a_{12} = 0$: monetary policy does not respond to the current output growth, since the policy maker does not observe the contemporaneous output (Sims and Zha (2006)).

(A2) $a_{23} = 0$: the production sector does not respond to the changes in the contemporaneous nominal interest rate (Christiano, Eichenbaum, and Evans (2005))

(A3) The monetary policy has no effect on the long-run impact on output (Blanchard and Quah (1993)).

In addition to, or as an alternative set of these dogmatic zero equality restrictions, one may want to impose sign restrictions on the impulse responses, as considered in Canova and Nicolo (2002), Faust (1998), and Uhlig (2005)).

(A4) The response of commodity price index to a contractionary monetary policy shock is negative over a certain period.

(A5) The response of the federal fund rate to a contractionary monetary policy shock is positive over a certain period.

Suppose that we are not fully sure of the long-run money neutrality restriction (A3), and consider dropping it from the set of assumptions. If we do so, it can be shown that the reduced set of restrictions, (A1), (A2), (A4), and (A5), only set-identifies the structural parameters in the sense of Manski's partial identification (Manski (2003)). When we confront such situation, how do we draw statistical inference for the impulse responses?

The standard Bayesian procedure with a prior distribution available for the whole structural parameters yields the proper posterior distribution, regardless of whether the imposed restrictions guarantee identification or not. This, however, does not mean that the Bayesian approach is free from the identification issue. When the structural parameters are not identified, the posterior distribution for the impulse responses remains to be sensitive to a choice of prior, even asymptotically (Kadane (1974), Poirier (1998), and Moon and Shorfheide (2012)), and the posterior distribution converges to a conditional prior distribution given the reduced form parameters equal to their true value. Such non-diminishing prior influence may raise challenges to Bayesians, especially when they cannot confidently specify or cannot agree upon a prior belief for all the structural parameters in the form of a probability distribution.

In order to relieve such sensitivity concerns and anxieties that the Bayesians may experience in analyzing non-identified models, this paper considers a posterior inference procedure from the multiple prior robust Bayes perspective. Instead of forcing one to form a prior for the entire structural parameters, we propose to specify a set of priors as a prior input. The Bayes rule is applied to each prior in the class to form the class of posteriors, and the class of posteriors is summarized by reporting the ranges of posterior mean and posterior probability as the posterior varies over the class. The posterior bound analysis of this type has been considered in the statistics literatures of robust Bayes analysis and global sensitivity analysis including Berger and Berliner (1986), DeRobertis and Hartigan (1981), and Wasserman (1990). In econometrics, Chamberlain and Leamer (1976) and Leamer (1982) pioneered the bound analysis of the posterior mean in the linear regression analysis. More recently, Kitagawa (2012) interprets the Manski's partial identification analysis from this robust Bayes viewpoint, and examines relationships between the range of posterior quantities and frequentist inference for the identified set. With the class of priors considered in this paper, the range of posterior quantities can be interpreted as summarizing the "posterior distribution of the impulse response identified *set*", and presenting the range of posterior quantities can also serve to draw a posterior inference for the identified set, as considered in Klein and Tamer (2013) and Liao and Simoni (2013).

The prior class considered in this paper consists of any priors that share a common prior for the reduced form parameters and satisfy the imposed zero and sign restrictions with probability one. When the imposed restrictions only set-identify the structural parameters, having a single prior for the reduced-form parameters cannot yield the single posterior for the structural parameters, and one must introduce a prior distribution to the non-identified components in the model in order to perform the standard Bayesian inference on the basis of a single posterior. In the context of the structural VAR with the sign restrictions, the non-identified component of the model corresponds to a rotation matrix that determines how the reduced form errors are decomposed to the structural shocks, and putting a prior for it corresponds to a common practice of introducing a prior distribution (typically uniform distribution) for the rotation matrices (Uhlig (2005)). Despite that this "agnostic" Bayesian analysis has been applied to a wide range empirical studies, this approach is subject to some criticism because the seemingly non-informative prior introduced for the non-identified part of the model leads to unintentionally informative prior for the impulse responses, and it may significantly influence the posterior analysis no matter how large the sample size is. This paper proposes an alternative to this widely-used, but controversial, "agnostic" Bayesian practice, by translating "agnostic" prior belief to "ambiguous" belief in the sense of multiple priors. Specifically, instead of discussing what prior for the rotation matrices represents "agnostic" prior knowledge, we advocate to admit arbitrary priors specified for them in the form of a class of priors, and to report the posterior inferential statement that is valid irrespective of what prior is used in the class. Since data (likelihood) are informative only for the reduced form parameters and not at all for the rotation matrix, our proposal of reporting the range of posterior quantities can be seen as reporting the posterior information for the object of interest based only on the well-updated part of belief, or synonymously, based only on the shape of the likelihood if the prior for the reduced form parameters is diffuse. This way of summarizing "what data (likelihood) says" is in a similar spirit to Manski's partial identification analysis, and extends the view of statistical inference based on the sample likelihood, as advocated in Sims (1998)[1], to a non-identified model.

As opposed to the agnostic Bayesian approach with a "noninformative" prior, Baumeister and Hamilton (2013), Fry and Pagan (2011), and Gordon and Boccanfuso (2001) insist importance of making an effort for eliciting a prior input on the basis of the researcher's belief on the structural objects. This paper does not intend to be against a use of a carefully elicited prior for the structural parameters, as far as the researcher can come up with it. Nevertheless, even when the researcher can confidently specify a prior for the structural parameters, our posterior bounds analysis is still useful and worth being reported, since it serves to summarize and visualize to what extent availability of the specific credible prior enhances informativeness of the posterior inference.

If we ignore the difference in the source of probability in inferential statements, the posterior bound analysis of this paper is similar to the frequentist inference procedures for the impulse response identified set proposed by Moon, Schorfheide, and Granziela (2013). Both approaches aim to draw inferential statement for impulse responses without relying on assumptions/belief other

[1] Sims and Zha (1998, pp1115) write "Reporting is not decision-making, and therefore makes no use of subjective prior beliefs .... Scientific reporting is then just the problem of conveying the shape of the likelihood to potencial users of the analysis."

than the dogmatic equality restrictions and/or the sign restrictions. They are, in fact, expected to give a similar result, at least asymptotically, since the posterior bound and the frequentist confidence intervals both converges to the true identified set. Whereas, in terms of implementation and the scope of applications, the posterior bounds analysis proposed in this paper differs from the frequentist approach of Moon, Shorfheide, and Granziera (2013) in several aspects. First, the posterior bound approach proposed in this paper is computationally less demanding than the construction of the frequentist confidence intervals since the construction of the posterior bounds avoids a construction of a non-rejection region of a test, which typically involves resampling-based computation of a critical value of some test statistics. Accordingly, the posterior bound analysis can be implemented straightforwardly even when the sign restrictions are placed for the impulse responses to multiple shocks. Also, it is straightforward to accommodate a wider class of zero restrictions than those considered in Moon, Shorfheide, and Granziera (2013). Second, the posterior bound analysis allows us to separately report the posterior belief for the *plausibility* of the imposed assumptions and the posterior belief for the impulse responses conditional on that the imposed assumptions are plausible, i.e., the distribution of data (value of the reduced form parameters) is consistent with the imposed restrictions. In contrast to our robust Bayes proposal, frequentist inference is generally difficult to separate out these two distinct sample information, inference for the object of interest and the measure of fitness (see Sims (1998) for enlightening discussion on this). Third, the posterior bound analysis does not involve any asymptotic approximations, so we believe that the proposed posterior inference procedure can be an attractive alternative for frequentists as well, especially, those who concern with the shape of the observed likelihood and is anxious about accuracy of the asymptotic approximations.

The remainder of the paper is organized as follows. Section 2 presents an illustrating example to overview the main proposal of this paper. Section 3 introduces notations and a general analytical framework of SVARs with zero and/or sign restrictions. Section 4 derives the identified set of the impulse responses under the zero and sign restrictions. Section 5 introduces the class of priors and presents a numerical procedure to compute the bounds of posterior means and posterior probabilities. An empirical example is given in Section 6. Technical proofs are collected in Appendix.

2 Posterior Sensitivity and Posterior Bounds: An Illustrating Example

To highlight a motivation of this paper and to overview our proposal of the posterior bound analysis, this section illustrates posterior sensitivity of an impulse response function in a commonly used Bayesian procedure for SVAR with sign restrictions.

Consider the following four variable SVAR with two lags, where the vector of observables consists of a nominal interest rate i_t, real GDP y_t, inflation rate π_t, and real money balances m_t.

$$A_0 \begin{pmatrix} i_t \\ \Delta y_t \\ \pi_t \\ m_t \end{pmatrix} = a + \sum_{l=1}^{2} A_l \begin{pmatrix} i_{t-l} \\ \Delta y_{t-l} \\ \pi_{t-l} \\ m_{t-l} \end{pmatrix} + \epsilon_t,$$

where (a, A_0, A_1, A_2) are an intercept vector and matrices of structural parameters, and $\epsilon_t = (\epsilon_{i,t}, \epsilon_{\Delta y,t}, \epsilon_{\pi,t}, \epsilon_{m,t})'$ is a vector of structural shocks that is assumed to be independent of the past realizations of any variables and follow Gaussian with mean zeros and the variance-covariance standardized to the identity matrix. Here, the output variable is transformed to the first difference (GDP growth) to have the reduced form VAR invertible. We write the reduced-form VAR as

$$\begin{pmatrix} i_t \\ \Delta y_t \\ \pi_t \\ m_t \end{pmatrix} = b + \sum_{l=1}^{2} B_l \begin{pmatrix} i_{t-l} \\ \Delta y_{t-l} \\ \pi_{t-l} \\ m_{t-l} \end{pmatrix} + u_t, \quad u_t \sim \mathcal{N}(0, \Sigma),$$

where (b, B_1, B_2, Σ) are reduced-form parameters. Let an impulse response function of interest be the output response to a monetary policy shock, $\frac{\partial y_{t+h}}{\partial \epsilon_{i,t}}$, at horizon $h = 1$. The dataset we use are from Aruoba and Schorfheide (2011).

To draw a posterior inference with sign restrictions on impulse responses, we follow a widely used "agnostic" Bayesian approach proposed by Uhlig (2005). As a set of sign restrictions, consider imposing the following,

- the inflation response to a contractionary monetary policy shock is nonpositive for one quarterly period; $\frac{\partial \pi_{t+h}}{\partial \epsilon_{i,t}} \leq 0$ for $h = 0, 1$.
- the interest rate response to the contractionary monetary policy shock is nonnegative for one quarterly period; $\frac{\partial i_{t+h}}{\partial \epsilon_{i,t}} \geq 0$ for $h = 0, 1$.
- the responses of the real money balances to the contractionary monetary policy shock is nonpositive for one quarterly period; $\frac{\partial m_{t+h}}{\partial \epsilon_{i,t}} \geq 0$, for $h = 0, 1$.

In a common empirical practice with these sign restrictions, prior inputs one specifies are a prior distribution for the reduced-form parameters and a prior distribution for a unit-length 4×1 vector q. Here, q plays a role of pinning down down structural monetary policy shock ϵ_{it} based on a reduced form error vector u_t via $\epsilon_{it} = q' \Sigma_{tr}^{-1} u_t$, where Σ_{tr} is the lower-triangular Cholesky decomposition of the variance-covariance matrix Σ. Since the impulse response of output to a unit shock in ϵ_{it} at $h = 1$ is obtained by $(2, 1)$ element of $(B_1 + I)\, \Sigma_{tr} q$, the posterior of (B_1, Σ) and a "prior" for q induce the posterior distribution of the impulse response.[2] Uhlig's "agnostic" Bayesian approach

[2] Since the value of the likelihood for the structural parameters depends only through the reduced-form parameters, the prior for reduced-form parameters can be updated by data, while the prior for q conditional on the reduced form parameters is never be updated.

in particular recommends a use of the uniform distribution on the unit sphere truncated according to the sign restrictions as a prior for q. The uniform prior on the unit sphere (without truncation) can be obtained by transforming the standard multivariate normal distribution in the following way,

$$q = \frac{z}{\|z\|}, \quad z \sim \mathcal{N}(0, I_{4\times 4}). \tag{2.1}$$

With a prior for the reduced-form parameters set at their Jeffrey's prior, the solid curve in Figure 1 plots the posterior density and the posterior mean of $\frac{\partial y_{t+1}}{\partial \epsilon_{i,t}}$ obtained by the Uhlig's agnostic approach. The posterior is unimodal with the 90% highest posterior credible region [-.52, .72]. Note that the seemingly uninformative uniform prior for q does not generally yield a flat region for the posterior of the impulse response, despite the fact that the sign restrictions can only set-identify the impulse response (Moon, Shorfheide, and Granziera (2013)). Also, we cannot generally claim that the uniform prior for q leads to the least informative posterior for the impulse response. Obtaining such seemingly informative posterior is not only a finite sample phenomenon, but it also occurs asymptotically. The asymptotic posterior of the impulse response is fully determined by the conditional prior of q given the reduced form parameters set at their true value. It is important to note that the uniform prior for q does not yield a uniform distribution of the impulse response even without any sign restrictions, because transforming the uniform distribution on the unit sphere via $(B_1 + I)\,\Sigma_{tr} q$ and marginalizing it to one coordinate leads to a non-uniform distribution. The posterior of impulse response therefore converges to some non-uniform distribution, which results in obscuring the lack of identifying information in data. Having known how the posterior behaves in relation to the prior input, discussion on what prior for q can be justified as "non-informative" or "agnostic" is inherently controversial, and, if no additional credible prior knowledge other than the sign restrictions are available, a uniform consensus on what prior for q should be used seems difficult to attain.

When the model lacks identification, another notable feature in the posterior analysis is a posterior sensitivity to a choice of prior, especially, for the non-identified components in the model, i.e., a prior of q in the current context. To illustrate this posterior sensitivity, The dot-dashed and dashed curves in Figure 1 show posterior densities when the Uhlig's agnostic prior is perturbed in two different ways. The first perturbation introduces positive correlations (0.5) among z. The resulting posterior is drawn as the dot-dashed density. The second perturbation instead introduces negative correlations (-0.3) among z, and the resulting posterior is drawn as the dashed density. As is evident from Figure 1, the shape of the posterior changes considerably as the prior variance-covariance matrix for z changes. There are, indeed, (infinitely) many different ways to perturb the prior for q, and there may well be a prior for q that changes the posterior more drastically. Furthermore, given the fact that the shape of the asymptotic posterior is governed by a prior for q, the posterior sensitivity we observe in Figure 1 will not vanish even asymptotically.

The non-diminishing posterior sensitivity and the absence of the consensus on a non-informative

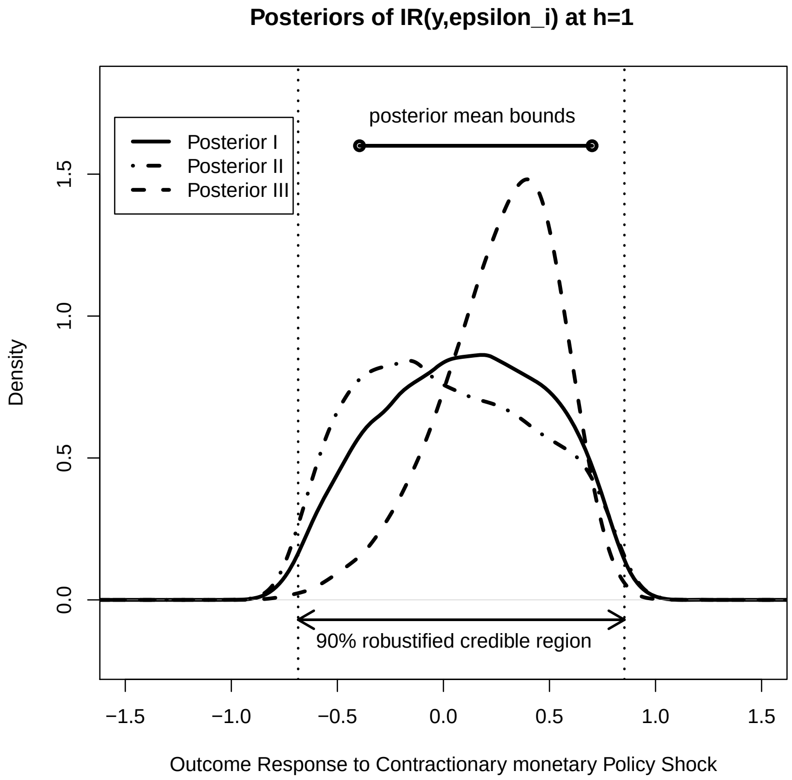

Figure 1: The prior for the reduced form parameters is fixed at the Jeffreys' prior $\propto |\Sigma|^{5/2}$. **Posterior 1 (solid)**: Uhlig's prior $q \sim z/\|z\|$, $z \sim \mathcal{N}(0, I_{4\times4})$. The posterior mean is .10. The highest posterior density region (HPD) is $[-.52, .72]$. **Posterior 2 (dot-dashed)**: prior $q \sim z/\|z\|$, $z \sim \mathcal{N}(0, \Omega_2)$ where $\Omega_2 = 0.5 \times I_{4\times4} + 0.5 \times \mathbf{11}'$. The posterior mean is .02. HPD is $[-.61, .69]$. **Posterior 3 (dashed)**: prior $q \sim z/\|z\|$, $z \sim \mathcal{N}(0, \Omega_3)$ where $\Omega_3 = 1.3 \times I_{4\times4} - 0.3 \times \mathbf{11}'$. The posterior mean is .26. HPD is $[-.19, .70]$. The **posterior mean bounds** are $[-.38, .70]$, and the **robustified credible region** with lower credibility 90% is $[-.66, .87]$.

prior for q pose a challenging question to Bayesians; what prior input for q is desirable when the available prior knowledge is exhausted by the set of sign and/or the insufficient number of zero restrictions? In this paper, we propose a posterior inference for the impulse responses by reporting the *range of posterior means and the posterior probabilities*, rather than discussing what choice of a prior for q is a most reasonable representation for the lack of prior knowledge on q. We develop a way to compute the range of posterior quantities when a prior for q is allowed to vary in an arbitrary way subject to the imposed sign and/or zero restrictions. For the current example, the posterior mean bounds are shown as the horizontal segment in Figure 1. We interpret the presented range of posterior means as that, *with keeping the prior for the reduced form parameters fixed, the posterior mean of the impulse response varies over this range as arbitrary priors for q are allowed.* Incapability of conveying which posterior means are more credible than the others is a honest and accurate description of the posterior knowledge in the absence of prior knowledge on q. The interval with arrows shows a robustified posterior credible region with credibility 90% as defined in Kitagawa (2012). The 90% robustified credible region shows *the shortest interval such that, with keeping the prior for the reduced form parameters fixed, the posterior probabilities on the interval are at least 90% irrespective of what prior for q is used.* The robustified credible regions can be plotted at each credibility level as in Figure 2, and they can be used to summarize and visualize the class of posterior distributions induced by multiple priors of q.

3 The Framework

Consider SVARs in the following general form

$$A_0 y_t = a + \sum_{l=1}^{p} A_l y_{t-l} + \epsilon_t \quad \text{for } t = 1, \ldots, T,$$

where y_t is an $n \times 1$ vector of endogenous variables, p the lag length, ϵ_t an $n \times 1$ vector of exogenous structural shocks independent of the past y_t's and ϵ_t's, A_l an $n \times n$ matrix of structural parameters for $l = 0, \ldots, p$, and a is a $n \times 1$ vector of intercepts. Assume that the distribution of structural shocks ϵ_t is n-variate Gaussian with mean zero and the variance-covariance matrix I_n, $n \times n$ identity matrix. The initial conditions, $y_1, \ldots, y_p$ are given.

We write the reduced form representation of this structural form as

$$y_t = b + \sum_{l=1}^{p} B_l y_{t-l} + u_t, \tag{3.1}$$

where $b = A_0^{-1} a$, $B_l = A_0^{-1} A_l$, $u_t = A_0^{-1} \epsilon_t$, and $E(u_t u_t') \equiv \Sigma = A_0^{-1} \left(A_0^{-1}\right)'$. We denote the reduced form coefficients by $B = [b, B_1, \ldots, B_p]$, and denote the entire reduced form parameters by $\phi = (B, \Sigma) \in \Phi \subset \mathcal{R}^{n+n^2 p} \times \Omega$, where Ω is the space of positive-semidefinite matrices. The value of likelihood depends only on ϕ, since the sampling distribution of data is fully characterized by ϕ. We set Φ the domain of ϕ to the set of (B, Σ) such that the reduced form VAR(p) can be inverted to VMA(∞).

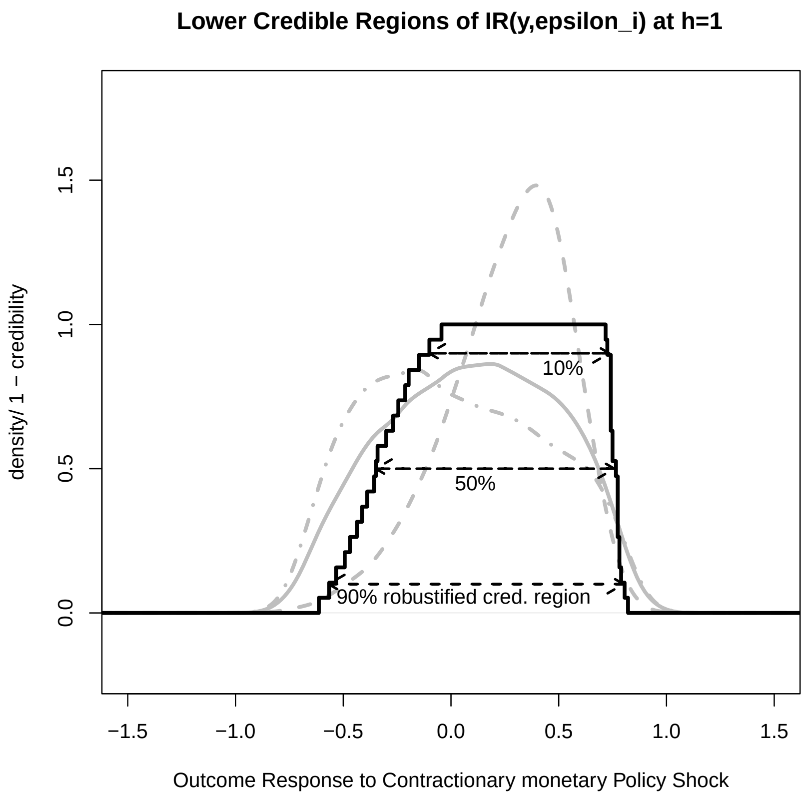

Figure 2: The step function summarizes the robustified credible regions at each credibility level, $\alpha = .95, .90, \ldots, .05$. The level set of the step function at level $y \in \{0.05, 0.10, \ldots 0.95\}$ corresponds to the robustified credible region at lower credibility level $\alpha = 1 - y$.

The impulse response functions are of central interest in inferring the dynamic causal effect of the structural shocks on the endogenous variables. We denote the h-th horizon impulse response matrix by an $n \times n$ matrix IR^h, $h = 0, 1, 2, \ldots$, where the (i, j)-element in IR^h gives the impulse response of i-th variable in y_{t+h} in response to a unit shock of the j-th structural shock in ϵ_t. An expression of IR^h can be obtained by the MA(∞) representation of the reduced form (3.1). If the reduced form lag polynomial $\left(I_n - \sum_{l=1}^{p} B_l L^p\right)$ is invertible, we have

$$
\begin{aligned}
y_t &= c + \sum_{l=0}^{\infty} C_l(B)\, u_{t-l} \\
&= c + \sum_{l=0}^{\infty} C_l(B)\, A_0^{-1} \epsilon_{t-l},
\end{aligned}
$$

where $C_l(B)$ is the l-th coefficient matrix of the inverted lag polynomial $\left(I_n - \sum_{l=1}^{p} B_l L^l\right)^{-1}$, which depends only on B. IR^h can be therefore written as

$$
IR^h = C_h(B)\, A_0^{-1}.
$$

The long-run impulse response matrix is defined as $IR^\infty = \lim_{h\to\infty} IR^h = \left(I_n - \sum_{l=1}^{p} B_l\right)^{-1} A_0^{-1}$, and the long-run cumulative impulse response matrix is defined as $CIR^\infty = \sum_{h=0}^{\infty} IR^h = \left(\sum_{h=0}^{\infty} C_h(B)\right) A_0^{-1}$.

Identification of the structural coefficients A_0 is essential in identifying the dynamic causal effects in the SVAR framework, while, in the absence of any restrictions, the knowledge of the reduced form parameters ϕ does not pin down a unique A_0. We can express the set of observationally equivalent A_0's given Σ using $Q \in \mathcal{O}(n)$ an $n \times n$ orthonormal matrix, where $\mathcal{O}(n)$ be the set of $n \times n$ orthonormal matrices. The individual column vectors in Q are denoted by $[q_1, q_2, \ldots, q_n]$. Denote the cholesky decomposition of Σ by $\Sigma = \Sigma_{tr}\Sigma_{tr}'$, where Σ_{tr} is the unique lower-triangular Cholesky factor with nonnegative diagonal elements. Note that any A_0 in the form of $A_0 = Q'\Sigma_{tr}^{-1}$ satisfies $\Sigma = (A_0' A_0)^{-1}$. So, in the absence of any identifying restrictions, $\left\{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n)\right\}$ forms the set of A_0's that are consistent with the reduced-form variance-covariance matrix Σ (Uhlig (2005) Proposition A.1). Since the likelihood function only depends on the reduced form parameters ϕ, data are silent about Q, which leads to ambiguity in decomposing Σ into the product of A_0^{-1}. If the imposed identifying restrictions fail to identify A_0, it means, for each given Σ, there are multiple Q's yielding the structural parameter matrix A_0 satisfying the imposed restrictions. See Rubio-Ramirez, Waggoner, and Zha (2010) for the definition of global identification of the structural parameters via the rotation matrix.

In the absence of any identifying restrictions on A_0, the only restrictions to be imposed for Q are the sign normalization restrictions for the structural shocks, which can be introduced by the sign restrictions on the diagonal elements of A_0. The sign normalization restriction for A_0 that we maintain throughout this paper is that the diagonal elements of A_0 are all positive. Depending on interpretation of the structural equations as well as the structural shocks, sign normalization restrictions may be imposed on different elements of A_0.

Once the sign normalization restrictions on A_0 are imposed, the set of observationally equivalent A_0's corresponding to Σ can be expressed as

$$\left\{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n), \quad diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}, \tag{3.2}$$

where the inequality restriction, $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$, means all the diagonal elements of $A_0 = Q'\Sigma_{tr}^{-1}$ are nonnegative. We express the sign normalization restriction in terms of the weak inequalities, so that the set of admissible rotation matrices Q satisfying $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$ is given as a closed set in $\mathcal{O}(n)$. By denoting the column vectors of Σ_{tr}^{-1} as $\left[\sigma^1, \sigma^2, \ldots, \sigma^n\right]$, this sign normalization restrictions can be written as a collection of linear inequalities,

$$q_i'\sigma^i \geq 0 \quad \text{for all } i = 1, \ldots, n.$$

We hereafter denote the set of Q's satisfying $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$ by $\mathcal{Q}(\phi)$, Σ-dependent closed subset in $\mathcal{O}(n)$.

Consider that we want to draw inference for an impulse response, say, $(\tilde{\imath}, j)$-element of IR^h,

$$r_{\tilde{\imath}j}^h \equiv e_{\tilde{\imath}}'C_h(B)\Sigma_{tr}Qe_j \equiv c_{\tilde{\imath}h}'(\phi)\, q_j,$$

where $e_{\tilde{\imath}}$ is $\tilde{\imath}$-th column vector of I_n and $c_{\tilde{\imath}h}'(\phi)$ is the $\tilde{\imath}$-th row vector of $C_h(B)\Sigma_{tr}$. To compress the notational complexity, we make $\tilde{\imath}$, j, and h in the subscripts and superscripts implicit in our notation unless confusions arises, and use $r \in \mathcal{R}$ to denote the impulse response of interest, i.e., $r \equiv r_{\tilde{\imath}j}^h$. When we want to emphasize the dependence of r on the reduced form parameters ϕ and the rotation matrix Q, we express r as $r(\phi, Q)$. Note that the identified set and the posterior bound analysis developed below cover not only impulse responses, but also structural parameters in A_0 and $[A_1, \ldots, A_p]$, since each structural parameter can be expressed by the inner product of a vector depending on ϕ and a column vector of Q, e.g., $(j, \tilde{\imath})$-entry of A_l can be expressed by $e_{\tilde{\imath}}'\left(\Sigma_{tr}^{-1}B_l\right)' q_j$.

4 Partially-Identified VARs

In what follows, when we say "... holds for almost all $\phi \in \Phi$," it means that "... holds for all $\phi \in \Phi$, except for the null set of ϕ in terms of the Lebesgue measure on Φ."

4.1 Identified Set with Under-identifying Zero Restrictions

In order to have informative estimation and inference for the response, the SVAR analysis imposes a set of restrictions that constrain the values of structural parameters. For instance, the triangularity restriction of A_0 with sign normalizations on the diagonal elements, which comes from the assumptions on contemporaneous causal ordering among the endogenous variables, unambiguously sets the lower-triangular components of A_0 equal to zeros. It has been also considered to impose equality restrictions setting some elements of the lagged structural coefficients, $\{A_l : l = 1, \ldots p\}$ equal to zeros. As another class of identifying restrictions, the empirical macroeconomics literature sometimes imposes restrictions on the long-run impulse responses; setting some elements in the long-run impulse response $IR^{\infty} = \left(I - \sum_{l=1}^{p} B_l\right)^{-1} \Sigma_{tr} Q$, or the long-run cumulative impulse response, $CIR^{\infty} = \sum_{h=0}^{\infty} C_h(B) \Sigma_{tr} Q$, equal to zeros (Blanchard and Quah (1993) among others). We refer to the equality restriction setting an element of the structural parameter matrix, or the impulse response matrices equal to zero as a *zero restriction*.

A collection of these zero restrictions can be represented in the following form,

$$F(\phi, Q) \equiv \begin{pmatrix} F_1(\phi) q_1 \\ F_2(\phi) q_2 \\ \vdots \\ F_{\tilde{n}}(\phi) q_{\tilde{n}} \end{pmatrix} = \mathbf{0}, \tag{4.1}$$

where $\tilde{n} \leq n$, $F_i(\phi)$ is an $f_i \times n$ matrix that can depend only on the reduced form parameters $\phi = (B, \Sigma)$. If $\tilde{n} < n$, then, for $i = (\tilde{n}+1), \ldots, n$, no zero restrictions are imposed for the corresponding column vectors of Q. Without loss of generality, we order the endogenous variables in such way that $f_1 \geq f_2 \geq \cdots \geq f_{\tilde{n}} > 0$ holds, and for $i = \tilde{n}+1, \ldots, n$, we define $f_i = 0$ just for expositional convenience. This form of zero restrictions can accommodate any zero restrictions that set elements of A_0, $\{A_l : l = 1, \ldots p\}$, IR^{∞}, and CIR^{∞} to zeros. For instance, by noting that $A_0' = \left(\Sigma_{tr}^{-1}\right)' Q$, setting (i,j)-element of A_0' equal to zero can be written as a linear equality constraint for j-th column vector of Q,

$$e_i' \left(\Sigma_{tr}^{-1}\right)' q_j = 0.$$

Similarly, by noting $A_l' = B_l' \left(\Sigma_{tr}^{-1}\right)' Q$, the equality restriction such that (i,j)-element of A_l' equals to zero can be written as,

$$e_i' \left(\Sigma_{tr}^{-1} B_l\right)' q_j = 0.$$

As for the lung-run impulse responses, $IR^{\infty} = \left(I - \sum_{l=1}^{p} B_l\right)^{-1} \Sigma_{tr} Q$, setting the (i,j)-element of IR^{∞} equal to zero can be also written as a linear equality constraint for j-th column vector of Q,

$$e_i' \left(I - \sum_{l=1}^{p} B_l\right)^{-1} \Sigma_{tr} q_j = 0.$$

$F_j(\phi)$ is defined accordingly by stacking the row vectors multiplied to q_j in these zero restrictions into a matrix.

A class of non-identified models considered in this paper consists of those, where *exact-identification* for the structural parameters fails in a certain manner. Following Definition 5 of Rubio-Ramirez et al (2010), the structural VAR is exactly identified if, for almost every $\phi \in \Phi$, there exists unique structural parameters $(A_0, A_1, \ldots, A_p)$ satisfying the imposed identifying restrictions. This can be equivalently said as that, for almost every $\phi \in \Phi$, there is unique Q satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations. By Lemma 9 of Rubio-Ramirez, Waggoner, and Zha (2010), a necessary condition for exact identification is, in our notation, given by that $f_i = n - i$ holds for all $i = 1, \ldots, n$.[3] This paper focuses on cases, where the number of zero restrictions imposed for each q_i is smaller than or equal to the one necessary for exact identification,

$$f_i \leq n - i \text{ for all } i = 1, \ldots, n, \tag{4.2}$$

If the inequalities in (4.2) are strict for some $i \in \{1, \ldots, n\}$, we call the model is *under-identified.* Note that the class of under-identified models considered here do not exhaust all the non-identified structural VARs, since there exists a model that does not fall into (4.2), but still the structural parameters are not globally identified for some reduced form parameter values with a positive measure. For instance, the example given in Section 4.4 of Rubio-Ramirez, Waggoner, and Zha (2010) provides an example with $n = 3$ and $f_1 = f_2 = f_3 = 1$, where local identification of the structural parameters is met, while their global identification fails. We leave such locally-identified, but not globally-identified models out of scope of this paper's analysis.

If the model is under-identified in the sense defined above, there exist multiple Q's satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations at almost every value of ϕ. Existence of multiple admissible Q's, given ϕ, can generate a collection of the impulse responses that are consistent with the reduced form parameter ϕ, and this collection constitutes the identified set of the impulse responses corresponding to ϕ. More formally, let us denote by $\mathcal{Q}(\phi|F)$ the set of Q's that meets the imposed restrictions (4.1) and the sign normalization given ϕ, $\mathcal{Q}(\phi|F) = \left\{Q \in \mathcal{O}(n) : F(\phi, Q) = \mathbf{0},\, diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}$. The identified set for r is defined as a set-valued map from ϕ to a subset in $\mathcal{R}$ that gives the range of $r(\phi, Q)$ when Q varies over its domain $\mathcal{Q}(\phi|F)$,

$$IS_r(\phi|F) = \{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F)\}.$$

This way of defining the identified set as a set-valued map of the reduced form parameters is the same as the one considered in Moon et al (2013).

The lemma given below demonstrates that $IS_r(\phi|F)$ is generally convex, and it can become ϕ-a.s. singleton if the inequalities of (4.2) hold with equalities at each $i \leq j$. In Lemma 4.1(ii) below, we state the point-identification condition for r with employing an algorithm developed by Rubio-Ramirez, Waggoner, and Zha (2010) that successively finds the orthonormal vectors consistent with the imposed zero restrictions.

Lemma 4.1 *Consider the zero restrictions of the form given by (4.1), where the order of variables is consistent with $f_1 \geq f_2 \geq \cdots \geq f_{\tilde{n}} > 0$. Assume $f_i \leq n - i$ holds for all $i = 1, 2, \ldots, n$.*

(i) The identified set for $r = c'_{\tilde{\imath}h}(\phi)\, q_j$ is non-empty, bounded, and convex for every $\tilde{\imath} \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots,$ at every $\phi \in \Phi$.

[3] By Theorem 7 of Rubio-Ramirez et al (2010), this necessary condition for exact-identification becomes a necessary and sufficient condition under additional regularity conditions.

(ii) Let $\phi \in \Phi$ be given. Assume that $f_i = n - i$ and $rank(F_i(\phi)) = f_i$ hold for all $i = 1, \ldots, j$. Consider an algorithm developed by Rubio-Ramirez et al (2010), which successively constructs the orthonormal vectors $q_1, \ldots, q_j$, consistent with the zero restrictions.

(Step 1) Let q_1 be a unit length vector satisfying $F_1(\phi)q_1 = 0$, which is unique up to sign since $rank(F_1(\phi)) = n - 1$ by the assumption.

(Step 2) Given q_1 of Step 1, find orthonormal vectors $q_2, \ldots, q_j$, by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} q_i = 0,$$

successively for $i = 2, 3, \ldots, j$. If

$$rank \begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} = n - 1 \text{ at each } i = 2, \ldots, j, \tag{4.3}$$

and q_j obtained by the algorithm satisfies $q_j'\sigma^j \neq 0$, then the identified set for $r = c_{\tilde{\imath}h}'(\phi)\, q_j$ is a singleton for every $\tilde{\imath} \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$. If the claim holds for almost every ϕ, the identified set for r becomes ϕ-a.s. a singleton, so that the impulse response functions to the j-th structural shock are point-identified.

Proof. See Appendix A. ■

Lemma 4.1 (i) shows that the identified set for the impulse responses under the zero restrictions is always convex. It also clarifies that the identified set of the impulse response never becomes empty at every variable and horizon, so any zero restrictions that can be represented in the form of (4.2) cannot be refuted by data. The convexity of the identified sets plays an important role in simplifying computation of the range of posterior probabilities, as discussed in the next section. Lemma 4.1 (ii) offers an algorithmic way of examining point-identification of the impulse response of interest. This lemma is a straightforward extension of Lemma 7 of Rubio-Ramirez, Waggoner, and Zha (2010), to the case where only a unique determination of the j-th column vector of Q is concerned. Whether the rank conditions (4.3) hold or not depends on the choice of zero restrictions and a value of reduced form parameters.[4] Our posterior bound analysis given below can apply to both set-identified and point-identified cases (in the sense stated in Lemma 4.2(ii)) without explicitly

[4] A situation where the rank conditions of Lemma 3.1 (ii) are guaranteed at almost every ϕ arises if the row vectors of $F_i(\phi)$ are spanned by the row vectors of $F_{i-1}(\phi)$ for all $i = 2, \ldots, j$. This condition holds when we impose the triangularity restrictions on A_0.

checking the rank condition (4.3). If the posterior probability (in terms of ϕ) of having the rank conditions (4.3) satisfied is one, then the bounds for the posterior of r collapses to a point, resulting in reporting a single posterior distribution.

This lemma ensures that, when the model is under-identified, the identified-set $IS_r(\phi|F)$ is given by a bounded convex interval. The posterior inference procedure and its robust Bayes interpretation to be proposed below are valid regardless of whether $IS_r(\phi|F)$ is convex or not, whereas its large-sample consistency property to the true identified set will rely on convexity of $IS_r(\phi|F)$ at the true ϕ. For this reason, it is useful to have the almost sure convexity result for the identified set.

To illustrate our framework and notations, we provide a couple of examples.

Example 4.1 *Consider a four variable SVAR ($n = 4$). Assume zero restrictions are placed for A_0 and IR^∞ in the following way,*

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{24} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \qquad IR^\infty = \begin{pmatrix} r_{11}^\infty & 0 & r_{13}^\infty & r_{14}^\infty \\ 0 & r_{22}^\infty & r_{23}^\infty & r_{24}^\infty \\ r_{31}^\infty & r_{32}^\infty & r_{33}^\infty & r_{34}^\infty \\ r_{41}^\infty & r_{42}^\infty & r_{43}^\infty & r_{44}^\infty \end{pmatrix}. \tag{4.4}$$

By recalling $A_0' = \left(\Sigma_{tr}^{-1}\right)' Q$ and $IR^\infty = \left(I_n - \sum_{l=1}^p B_l\right)^{-1} \Sigma_{tr} Q$, these restrictions can be written as

$$\begin{aligned} e_4' A_0' e_1 &= e_4' \left(\Sigma_{tr}^{-1}\right)' q_1 = 0, \\ e_4' A_0' e_3 &= e_4' \left(\Sigma_{tr}^{-1}\right)' q_3 = 0, \\ e_1' \, IR^\infty e_2 &= e_1' \left(I_n - \sum_{l=1}^p B_l\right)^{-1} \Sigma_{tr} q_2 = 0, \\ e_2' \, IR^\infty e_1 &= e_2' \left(I_n - \sum_{l=1}^p B_l\right)^{-1} \Sigma_{tr} q_1 = 0. \end{aligned}$$

By collecting and sorting these restrictions for each q_i, we obtain the coefficient matrices of (4.1) as

$$\begin{aligned} F_1(\phi) &= \begin{pmatrix} e_4' \left(\Sigma_{tr}^{-1}\right)' \\ e_2' \left(I_n - \sum_{l=1}^p B_l\right)^{-1} \Sigma_{tr} \end{pmatrix}, \quad F_2(\phi) = e_1' \left(I_n - \sum_{l=1}^p B_l\right)^{-1} \Sigma_{tr}, \\ F_3(\phi) &= e_4' \left(\Sigma_{tr}^{-1}\right)', \end{aligned}$$

and $f_1 = 2$, $f_2 = 1$, and $f_3 = 1$. This model is under exactly-identified since $f_1 = 2 < 4 - 1$ and $f_2 = 1 < 4 - 2$. Suppose we are interested in responses to a structural shock in the third variable. They are not point-identified since the conditions of Lemma 4.1 (ii) fail for $j = 3$ at every value of ϕ. Whereas, Lemma 4.1 (i) guarantees that the identified sets of each impulse response is non-empty and convex at every horizon.

Example 4.2 *Consider the illustrating example given in Introduction. Suppose we impose (A1) and (A2) only. With the ordering of variables unchanged, these identifying restrictions can be written as*

$$\begin{aligned} F_1(\phi) q_1 &= e_2' \left(\Sigma_{tr}^{-1}\right)' q_1 = 0, \\ F_2(\phi) q_2 &= e_3' \left(\Sigma_{tr}^{-1}\right)' q_2 = 0, \end{aligned}$$

that is, $f_1 = f_2 = 1$. Suppose the impulse response of interest is the impact of monetary policy shock (ϵ_{Mt}) on output, e.g., $j = 1$. Again, point-identification of this impulse response fails since the conditions of Lemma 4.1 (ii) are not met for any value of ϕ.

4.2 Sign Restrictions on Impulse Response

It is straightforward to incorporate the sign restrictions on the impulse responses into the current framework. Adding sign restrictions of the impulse responses to the zero restrictions indeed tightens up the identified set of the impulse responses. Given the zero restrictions $F(\phi, Q) = \mathbf{0}$, we maintain the order of variables as in the previous section, i.e., $f_1 \geq f_2 \geq \cdots \geq f_{\tilde{n}} > 0$ holds. In case there are no zero restrictions and only the sign restrictions are imposed, we let the order of variables arbitrary, and $f_1 = \cdots = f_n = 0$ hold. Consider sign restrictions placed on the responses to the i-th structural shock. Suppose that, on the h-th horizon impulse responses, we impose $s_h \leq n$ number of sign restrictions. Since the impulse response vector to the i-th structural shock is given by the i-th column vector of $IR^h = C_h(B) \Sigma_{tr} Q$, we can write the sign restrictions placed on the h-th horizon response vector as,

$$S_{h,i}(\phi) q_i \geq \mathbf{0},$$

where the inequality is interpreted as the component-wise inequalities, $S_{h,i}(\phi) \equiv D_{h,i} C_h(B) \Sigma_{tr}$ is a $s_{h,i} \times n$ matrix, and $D_{h,i}$ is the $s_{h,i} \times n$ selection matrix that selects the sign restricted responses from the $n \times 1$ response vector $C_h(B) \Sigma_{tr} q_i$. Note that the nonzero elements of $D_{h,i}$ take 1 or -1 depending on whether the corresponding impulse responses are restricted to be positive or negative. By stacking the coefficient matrices $S_{h,i}(\phi)$ over the multiple horizons, we express the whole set of sign restrictions imposed on the responses to i-th shock by

$$S_i(\phi) q_i \geq \mathbf{0}, \tag{4.5}$$

where $S_i(\phi)$ is a $\left(\sum_{h=0}^{\bar{h}_i} s_{h,i}\right) \times n$ matrix defined by $S_i(\phi) = \left[S_{1,i}(\phi)', \ldots, S_{\bar{h}_i,i}(\phi)\right]'$. (if no sign restrictions are placed for the $\tilde{h}$-th horizon responses, $0 \leq \tilde{h} \leq \bar{h}$, we set $s_{\tilde{h},i} = 0$ and interpret $S_{\tilde{h},i}(\phi)$ is not present in the construction of $S_i(\phi)$.) Note that the sign restrictions considered here do not have to be restricted to the impulse responses. Since the matrices of structural parameters can be written as $A_0' = \Sigma_{tr}^{-1\prime} Q$ and $A_l' = B_l' \left(\Sigma_{tr}^{-1}\right)' Q$, $l = 1, \ldots, p$, any sign restrictions on structural parameters appearing in the i-th row of A_0 or A_l takes the form of linear inequalities for q_i as well, so these sign restrictions can be appended to $S_i(\phi)$ in (4.5).

Let $\mathcal{I}_S \subset \{1, 2, \ldots, n\}$ be the set of indices, such that $i \in \mathcal{I}_S$ if some of the impulse responses to i-th structural shock are sign-constrained, i.e., the set of sign constraints involve linear inequalities for q_i. The set of all the sign constraints can be accordingly expressed by

$$S_i(\phi)\, q_i \geq \mathbf{0} \quad \text{for } i \in \mathcal{I}_S. \tag{4.6}$$

As a shorthand notation, we represent the entire set of sign restrictions by $S(\phi, Q) \geq \mathbf{0}$.

Given $\phi \in \Phi$, let $\mathcal{Q}(\phi|F, S)$ be the set of Q's that jointly satisfy the sign restrictions (4.6), zero restrictions (4.1), and the sign normalizations, .

$$\mathcal{Q}(\phi|F, S) = \left\{Q \in \mathcal{O}(n) : S(\phi, Q) \geq \mathbf{0},\ F(\phi, Q) = \mathbf{0},\ diag\left(Q'\Sigma_{tr}^{-1}\right) \geq \mathbf{0}\right\}. \tag{4.7}$$

Being different from the case with only zero restrictions, $\mathcal{Q}(\phi|F, S)$ can be an empty set depending on ϕ and the imposed sign restrictions. If $\mathcal{Q}(\phi|F, S)$ is nonempty, the identified set of r denoted by $IS_r(\phi|F, S)$ is obtained by the range of r with the domain of Q given by $\mathcal{Q}(\phi|F, S)$. If $\mathcal{Q}(\phi|F, S)$ is empty, the identified set of r is defined as an empty set.

In contrast to the case with only the zero restrictions, $IS_r(\phi|F, S)$ is not always a convex set. The next lemma presents a sufficient condition for $IS_r(\phi|F, S)$ to be ϕ-a.s. convex.

Lemma 4.2 *Let $r = c_{\tilde{i}h}(\phi)\, q_j$ be an impulse response of interest. Suppose $\mathcal{I}_S = \{j\}$, i.e., the sign restrictions are placed only for the impulse responses to the j-th structural shock. If zero restrictions $F(\phi, Q) = 0$ satisfy either one of the following conditions,*

1. *the condition of Lemma 4.1 (ii) holds up to index $i = 1, \ldots, j,$, i.e., the zero restrictions pin down a unique $[q_1, q_2, \ldots, q_j]$, ϕ-a.s.*
2. *$j = 1$ and $f_1 \leq n - 1$, or $j \geq 2$ and $f_i < n - i$ holds for all $i = 1, \ldots, j - 1$,*
3. *$j \geq 2$ and there exists index $i^* \in \{1, \ldots, j - 1\}$ such that the condition of Lemma 4.1 (ii) holds up to index $i = 1, \ldots, i^*$, and $f_i < n - i$ holds for all $i = i^* + 1, i^* + 2, \ldots, (j - 1)$ if $i^* + 1 \leq j - 1$.*

then $IS_r(\phi|F, S)$ is convex for every $\tilde{i} \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots,$ whenever $IS_r(\phi|F, S)$ is nonempty.

Proof. See Appendix A. ■

Lemma B.1 of Moon, Shorfheide, and Granziera (2013) shows convexity of the impulse response identified set for the case where $\mathcal{I}_S = \{j\}$ and zero restrictions are imposed only for q_j. This lemma extends their lemma to the cases in which we can have zero restrictions on the column vectors of

Q other than q_j. In case that sign restrictions are imposed for impulse responses to a structural shock other than j-th shock, i.e., $\mathcal{I}_S$ contains an index other than j, we can find an example in which the identified set for an impulse response becomes non-convex.[5] Note that the conditions for zero restrictions $F(\phi, Q) = \mathbf{0}$ imposed in Lemma 4.2 are stronger than the one given in Lemma 4.1. We leave for future research further investigation on whether these conditions for $F(\phi, Q) = \mathbf{0}$ in Lemma 4.2 can be weakened or not.

Example 4.3 *To illustrate the notations introduced in this section, consider again the three variable example given in Introduction. Suppose we impose sign restrictions (A4) and (A5) for $h = 0, 1, \ldots, 5$. Then, the sign restrictions at horizon $h \in \{0, 1, , \ldots, 5\}$ are written as*

$$\begin{pmatrix} -c'_{2h}(\phi) \\ c'_{3h}(\phi) \end{pmatrix} q_1 \geq 0,$$

where $c'_{kh}(\phi)$ is the k-th row vector of $C_h(B)\Sigma_{tr}$. Hence, $S_{h,1}(\phi) = \begin{pmatrix} -c'_{2h}(\phi) \\ c'_{3h}(\phi) \end{pmatrix}$, $h = 0, 1, \ldots, 5$. Stacking $\{S_{h,1}(\phi) : h = 0, 1, \ldots, 5\}$ yields 12×3 matrix $S_1(\phi)$, and the whole sign restrictions is expressed as $S_1(\phi) q_1 \geq \mathbf{0}$.

5 Multiple Priors and the Posterior Bounds for Impulse Responses

5.1 Posterior Bounds: Analytical Representation

Let $\tilde{\pi}_\phi$ be a probability measure on the reduced form parameter space Φ. To construct a prior distribution for ϕ consistent with the zero restrictions $F(\phi, Q) = \mathbf{0}$ and sign restrictions $S(\phi, Q) \geq \mathbf{0}$, we trim the support of $\tilde{\pi}_\phi$ as follows,

$$\pi_\phi \equiv \tilde{\pi}_{\phi|\Phi_{F,S}} \equiv \frac{\tilde{\pi}_\phi 1\{\mathcal{Q}(\phi|F,S) \neq \emptyset\}}{\tilde{\pi}_\phi(\{\mathcal{Q}(\phi|F,S) \neq \emptyset\})},$$

where the conditioning event $\Phi_{F,S}$ in the notation of $\tilde{\pi}_{\phi|\Phi_{F,S}}$ is the set of reduced form parameter values that is consistent with the imposed restrictions, $\Phi_{F,S} = \{\phi \in \Phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}$. By construction, prior π_ϕ assigns probability one to the distribution of data that is consistent with the zero restrictions and the sign restrictions, i.e., $\pi_\phi(\{\mathcal{Q}(\phi|F,S) \neq \emptyset\}) = 1$. A joint prior for $(\phi, Q) \in \Phi \times \mathcal{O}(n)$ that has ϕ-marginal π_ϕ can be expressed as $\pi_{\phi,Q} = \pi_{Q|\phi}\pi_\phi$, where $\pi_{Q|\phi}$ is supported only on $\mathcal{Q}(\phi|F,S) \subset \mathcal{O}(n)$. Since the structural parameters $(A_0, A_1, \ldots, A_p)$ and any impulse responses

[5] Consider the example given in Section 4.4 of Rubio-Ramirez (2010), where $n = 3$ and their zero restrictions satisfy $f_1 = f_2 = f_3 = 1$. They show that the identified set of an impulse response function consists of two distinct points. If we interpret the zero restrictions placed for the second and the third variables as pairs of linear inequality restrictions for q_2 and q_3 with opposite signs, convexity of $IS_r(\phi|F,S)$ fails. In this counterexample, the assumption of $\mathcal{I}_S = \{j\}$ fails.

are functions of (ϕ, Q), $\pi_{\phi,Q}$ induces the unique prior distribution for $(A_0, A_1, \ldots, A_p)$ as well as the prior for the impulse responses. In turn, if we specify a prior distribution for $(A_0, A_1, \ldots, A_p)$ with incorporating the sign normalizations, it induces a prior for $\pi_{\phi Q}$. If one implements a standard Bayesian inference with having $\pi_{\phi,Q}$ or a prior for $(A_0, A_1, \ldots, A_p)$ as a prior input, the data only allow us to update the marginal prior for ϕ, whereas the conditional prior $\pi_{Q|\phi}$ is never be updated by data, since Q does not appear in the likelihood, i.e., Q is conditionally independent of data given ϕ. This means that, if the analyst conducts SVAR analysis with a prior distribution for $(A_0, A_1, \ldots, A_p)$, the prior for ϕ induced by the prior of $(A_0, A_1, \ldots, A_p)$ is well-updated by data, while the conditional prior $\pi_{Q|\phi}$, which is implicitly induced by the prior of $(A_0, A_1, \ldots, A_p)$, remains as it is.

In the exact identification case where the imposed restrictions and the sign normalizations can pin down a unique Q ($\mathcal{Q}(\phi|F, S)$ is a singleton) at π_ϕ-almost every ϕ, $\pi_{Q|\phi}$ is degenerate and gives a point mass at such Q. Accordingly, with the point-identifying restrictions a priori imposed in a dogmatic way, specifying π_ϕ suffices to induce the single posterior distribution for the structural coefficients as well as all the impulse responses. In contrast, in the partially identified situation where $\mathcal{Q}(\phi|F, S)$ is non-singleton for ϕ's with a positive measure, specifying solely π_ϕ cannot yield a unique posterior distribution for the impulse responses. To have a posterior distribution for the impulse responses, as desired in the standard Bayesian approach, we need to specify $\pi_{Q|\phi}$, which is supported only on $\mathcal{Q}(\phi|F, S) \subset \mathcal{O}(n)$ at each $\phi \in \Phi$. In empirical practice, however, it is a challenging task for a researcher to come up with a "reasonable" specification for $\pi_{Q|\phi}$ especially when his prior knowledge that he considers credible is exhausted by the zero restrictions and the sign restrictions $S(\phi, Q) \geq 0$. Even when it is feasible to specify $\pi_{Q|\phi}$, the fact that $\pi_{Q|\phi}$ is never be updated by data makes the posterior distributions for the impulse responses remain to be sensitive to the choice of $\pi_{Q|\phi}$ even asymptotically, so that a limited confidence in the choice of $\pi_{Q|\phi}$ leads to an equally limited credibility in the posterior inference. Since (ϕ, Q) and the structural parameters $(A_0, A_1, \ldots, A_p)$ are one-to-one (under the sign normalizations), the difficulty of specifying a prior for $\pi_{Q|\phi}$ can be equivalently stated as the difficulty of specifying a joint prior for the whole structural parameters with fixing the prior for ϕ at π_ϕ.

The robust Bayes procedure considered in this paper aims to make the posterior inference free from a choice of $\pi_{Q|\phi}$. More specifically, we specify a single prior for the reduced form parameters ϕ, which the likelihood are always informative about, while, instead of discussing what is a desirable choice of $\pi_{Q|\phi}$ nor seeking for a consensus on what is a least informative $\pi_{Q|\phi}$, we introduce a set of priors (ambiguous belief) for the conditional prior of Q given ϕ. Let $\Pi_{Q|\phi}$ denote a collection of conditional priors $\pi_{Q|\phi}$. Given a single prior for ϕ, π_ϕ, let $\pi_{\phi|Y}$ be the posterior distribution for ϕ obtained by the Bayesian reduced-form VAR, where Y stands for a sample. The posterior of ϕ combined with the prior class $\Pi_{Q|\phi}$ generates the class of joint posteriors of (ϕ, Q),

$$\Pi_{\phi Q|Y} = \left\{ \pi_{\phi Q|Y} = \pi_{Q|\phi}\pi_{\phi|Y} : \pi_{Q|\phi} \in \Pi_{Q|\phi} \right\},$$

which coincides with the class of posteriors obtained by applying the Bayes rule to each prior in the class $\{\pi_{\phi,Q} = \pi_\phi \pi_{Q|\phi} : \pi_{Q|\phi} \in \Pi_{Q|\phi}\}$. This class of posteriors of (ϕ, Q) induces the class of posteriors for impulse response, $r = r(\phi, Q)$,

$$\Pi_{r|Y} \equiv \left\{ \pi_{r|Y}(\cdot) = \pi_{\phi,Q|Y}(r(\phi, Q) \in \cdot) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y} \right\}. \tag{5.1}$$

We summarize the posterior class of r by constructing the bounds of the posterior means of r and the posterior probabilities.

The class of conditional priors that puts no restrictions other than the zero restrictions or the sign restrictions is defined as

$$\Pi_{Q|\phi} = \left\{ \pi_{Q|\phi} : \pi_{Q|\phi}(\mathcal{Q}(\phi|F, S)) = 1,\ \pi_\phi\text{-almost surely} \right\}. \tag{5.2}$$

In words, this class consists of arbitrary $\pi_{Q|\phi}$'s as far as they assign probability one over Q's that meet the imposed restrictions. Kitagawa (2012) focuses on this type of prior class for a general class of partially identified models, and the first claim of the next proposition can be obtained as a corollary of Theorem 3.1 in Kitagawa (2012).[6]

Proposition 5.1 *Let a prior for ϕ, π_ϕ, be given, and assume $\pi_\phi(\{\phi : \mathcal{Q}(\phi|F, S) \neq \emptyset\}) = 1$. Let a prior class for $\pi_{Q|\phi}$ be given by (5.2).*

(i) The bounds of the posterior probabilities for an event $\{r \in G\}$, where G is a measurable subset in $\mathcal{R}$, are given by $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$, where

$$\begin{aligned} \pi_{r|Y*}(G) &\equiv \inf\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\ &= \pi_{\phi|Y}(IS_r(\phi|S, F) \subset G), \\ \pi^*_{r|Y}(G) &\equiv \sup\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\ &= \pi_{\phi|Y}(IS_r(\phi|S, F) \cap G \neq \emptyset), \\ &= 1 - \pi_{r|Y*}(G^c). \end{aligned}$$

(ii) The range of the posterior means $E(r|Y)$ with the posterior class $\Pi_{r|Y}$ given in (5.1) is

$$\left[\int_\Phi l(\phi)\, d\pi_{\phi|Y}, \int_\Phi u(\phi)\, d\pi_{\phi|Y}\right], \tag{5.3}$$

where $l(\phi)$ is the lower bound of $IS_r(\phi|F, S)$, $l(\phi) = \inf\{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F, S)\}$, and $u(\phi)$ is the upper bound of $IS_r(\phi|F, S)$, $u(\phi) = \sup\{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F, S)\}$.

[6] In our notation, (ϕ, Q) corresponds to the model parameters θ of Kitagawa's notation. Our notation for the reduced form parameters ϕ has the same meaning as the Kitagawa (2012)'s ϕ notation. The impulse response of interest $r^h_{ij} = c'_{ih}(\phi)q_j$ corresponds to the parameter of interest in Kitagawa's notation, $\eta(\theta) \in \mathcal{R}$.

Proof. The first claim is a corollary of Theorem 3.1 in Kitagawa (2012). For a proof of the second claim, see Appendix A. ■

Note that this proposition is valid irrespective of whether $IS_r(\phi|S,F)$ is a convex interval or not, so the formulas of the posterior probability bounds and the mean bounds apply to any set-identified SVARs. The presented posterior probability bounds are convex in the sense every value in $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$ is attained by some posterior in $\Pi_{\phi,Q|Y}$ (see Lemma B.1 of Kitagawa (2012) for a proof of this statement). As the expressions of the $\pi_{r|Y*}(G)$ and $\pi^*_{r|Y}(G)$ suggest, the bounds of the posterior probabilities can be computed by the posterior probability that G contains and intersects with the identified set of r, respectively. If the impulse response is point-identified in the sense of $IS_r(\phi|S,F)$ being $\pi_{\phi|Y}$-almost surely a singleton, the posterior probability bounds collapses to a point for every G, leading to the single posterior. We can approximate these posterior probability bounds if we can compute $IS_r(\phi|S,F)$ at values of ϕ randomly drawn from its posterior $\pi_{\phi|Y}$. Computation of $IS_r(\phi|S,F)$ can be greatly simplified if $IS_r(\phi|S,F)$ is guaranteed to be convex, e.g., the cases where Lemma 4.1 and Lemma 4.2 apply, since obtaining convex $IS_r(\phi|S,F)$ is reduced to computing $l(\phi)$ and $u(\phi)$.

The posterior mean bounds are given by the mean of the lower and upper bounds of $IS_r(\phi|S,F)$ taken with respect to the posterior of ϕ. The range of the posterior means are convex irrespective of whether the identified sets of r are convex or not.

Based on Proposition 4.1, our robust Bayes inference proposes to report the posterior mean bounds of (5.3). As a robustified credible region, we consider reporting an interval satisfying

$$\pi_{r|Y*}(C_{r,\alpha}) \geq \alpha. \tag{5.4}$$

$C_{r,\alpha}$ is interpreted as an interval estimate for r such that the posterior probability put on $C_{r,\alpha}$ is greater than or equal to α uniformly over the posteriors in the posterior class (5.1). There are multiple ways to construct $C_{r,\alpha}$ satisfying (5.4). One proposal is the one that has shortest width (Kitagawa (2012)) and meets (5.4) with equality. We hereafter refer to it as the *robustified credible region with lower credibility* α. We can also define $C_{r,\alpha}$ by mapping the highest posterior density region of ϕ to the real line via the set-valued map $IS_r(\cdot|S,F)$ (Moon and Schorfheide (2011)), which can be conservative in the sense that (5.4) can hold with inequality See also Klein and Tamer (2013) and Liao and Simoni (2013) for alternative ways to construct $C_{r,\alpha}$.

5.2 Computing Posterior Bounds

This subsection presents an algorithms to approximate the posterior quantities introduced in the previous subsection, using random draws of ϕ from its posterior.

Algorithm 5.1 *Let $F(\phi,Q) = \mathbf{0}$ and $S(\phi,Q) \geq \mathbf{0}$ be given, and let $r = c'_{ih}(\phi)\, q_j$ be an impulse response of interest.*

(Step 1) *Specify* $\tilde{\pi}_\phi$ *a prior for the reduced form parameters* ϕ. *The proposed* $\tilde{\pi}_\phi$ *need not satisfy* $\tilde{\pi}_\phi\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right) = 1$. *Run a Bayesian reduced form VAR to obtain the posterior* $\tilde{\pi}_{\phi|Y}$.

(Step 2) *Draw a reduced form parameter vector* ϕ *from* $\tilde{\pi}_{\phi|Y}$. *Given the draw of* ϕ, *we examine* $\mathcal{Q}(\phi|F,S)$ *is empty or not by following a subroutine (Step 2.1) - (Step 2.3) below.*

(Step 2.1) *Let* $z_1 \sim \mathcal{N}(0, I_n)$ *be a draw of an* n*-variate standard normal random variable. Let* $\mathcal{M}_1 z_1$ *be the* $n \times 1$ *residual vector in the linear projection of* z_1 *onto a* $n \times f_1$ *regressor matrix* $F_1(\phi)'$. *Set* $\tilde{q}_1 = \mathcal{M}_1 z_1$. *For* $i = 2, 3, \ldots, n$, *we run the following procedure sequentially; draw* $z_i \sim \mathcal{N}(0, I_n)$, *and compute* $\tilde{q}_i = \mathcal{M}_i z_i$, *where* $\mathcal{M}_i z_i$ *is the residual vector in the linear projection of* z_i *onto the* $n \times (f_i + i - 1)$ *regressor matrix,* $\left[F_i(\phi)', \tilde{q}_1, \ldots, \tilde{q}_{i-1}\right]$.

(Step 2.2) *Given* $\tilde{q}_1, \ldots, \tilde{q}_n$ *obtained in the previous step, define*

$$Q = \left[sign\left(\tilde{q}_1' \sigma^1\right) \frac{\tilde{q}_1}{\|\tilde{q}_1\|}, \ldots, sign\left(\tilde{q}_n' \sigma^n\right) \frac{\tilde{q}_n}{\|\tilde{q}_n\|} \right],$$

where $\|\cdot\|$ *is the Euclidian metric in* $\mathcal{R}^n$. *Thus-constructed* Q *can be seen as a draw of an orthogonal matrix from* $\mathcal{Q}(\phi|F)$.

(Step 2.3) *If* Q *obtained in (Step 2.2) satisfies the sign restrictions* $S(\phi, Q) \geq \mathbf{0}$, *retain this* Q *and proceed to (Step 3). Otherwise, repeat (Step 2.1) and (Step 2.2) at most* L *times (e.g.,* $L = 10000$*), until we obtain* Q *satisfying* $S(\phi, Q) \geq \mathbf{0}$. *If none of* L *number of draws of* Q *satisfies* $S(\phi, Q) \geq \mathbf{0}$, *we then approximate* $\mathcal{Q}(\phi|F,S)$ *to be empty, and go back to Step 2 to obtain a new draw of* ϕ.

(Step 3) *Given* ϕ *and* Q *obtained in (Step 2) and (Step 2.3), compute the lower and upper bound of* $IS_r(\phi|S,F)$ *by solving the following nonlinear optimizations with equality and inequality constraints,*[7]

$$\begin{aligned} & l(\phi) = \arg\min_Q c_{ih}'(\phi)\, q_j, \\ s.t. \quad & Q'Q = I_n, \quad F(\phi, Q) = \mathbf{0}, \\ & diag(Q'\Sigma_{tr}^{-1}) \geq 0, \text{ and } S(\phi, Q) \geq \mathbf{0}, \end{aligned}$$

and $u(\phi) = \arg\max_Q c_{ih}'(\phi)\, q_j$ *under the same set of constraints.*

(Step 4) *Repeat (Step 2) - (Step 3)* M *times, and obtain* M *draws of the intervals,* $[l(\phi_m), u(\phi_m)]$, $m = 1, \ldots, M$. *We then approximate the posterior mean bounds of Proposition 4.1 by the sample average of* $(l(\phi_m) : m = 1, \ldots, M)$ *and* $(u(\phi_m) : m = 1, \ldots, M)$.

[7] In the empirical application given in Section 5, we used "auglag" function available in an R package "alabama", which implements the augumentated Lagrangean multiplier method for a nonlinear optimization with equality and inequality constraints. At each ϕ, we use Q obtained in (Step 2.3) as an initial value for the nonlinear optimization. For all the models considered in Section 5, the optimization algorithm converged under the default convergence criterion at every draw of ϕ.

(Step 5) To construct the robustified credible region, define $d(r,\phi) = \max\{|r - l(\phi)|, |r - u(\phi)|\}$, and let $\hat{k}_\alpha(r)$ be the sample α-quantile of $(d(r,\phi_m) : m = 1,\ldots,M)$. The robustified credible region for r is obtained as an interval that centers at $\arg\min_r \hat{k}_\alpha(r)$ with radius $\min_r \hat{k}_\alpha(r)$ (Proposition 5.1 of Kitagawa (2012)).

In the above algorithm, the non-linear optimization part of (Step 3) can be computationally unstable and time-consuming, especially when the number of variables and constraints are large and convergence to the optimum is slow. If one comes up with such computational challenge in a given application, a more computationally stable algorithm can be used, in which (Step 3) above is replaced with (Step 3') below. A downside of this alternative algorithm is that the approximated identified set is smaller than $IS_r(\phi|F,S)$ at every draw of ϕ, resulting in approximating the posterior bounds to be shorter than the actual ones.

(Step 3') Iterate (Step 2.1) - (Step 2.3) L times and let $\left(Q_l : l = 1,\ldots,\tilde{L}\right)$ be those that satisfy the sign restrictions. (If none of the draws satisfies the sign restrictions, we draw new ϕ and iterate (Step 2.1) - (Step 2.3) again). Let $q_{j,l}$, $l = 1,\ldots,\tilde{L}$, be the j-th column vector of Q_l. We then approximate $[l(\phi), u(\phi)]$ by $[\min_l c'_{ih}(\phi) q_{j,l}, \max_l c'_{ih}(\phi) q_{j,l}]$.

5.3 Posterior Probability of the Empty Identified Set

By calculating the proportion of drawn ϕ's passing (Step 2.3) of Algorithm 4.1, we can obtain an approximation of the posterior probability (corresponding to the non-trimmed prior $\tilde{\pi}_\phi$) of having a nonempty identified set, $\tilde{\pi}_{\phi|Y}(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\})$. With the under-identifying zero restrictions only, the set of admissible Q's, $\mathcal{Q}(\phi|F)$, is never be empty, so that data never be able to detect violation of the imposed assumptions irrespective of a choice of $\tilde{\pi}_\phi$. In contrast, with the sign restrictions imposed, $\mathcal{Q}(\phi|F,S)$ can become empty for some ϕ, so that, if we specify $\tilde{\pi}_\phi$ that supports entire Φ (e.g., the normal -Wishart prior for $\phi = (B,\Sigma)$), data allow us to update the belief for *plausibility* of the imposed assumptions (i.e., belief for having a non-empty identified set). As is also claimed in Klein and Tamer (2013), we consider that the posterior *plausibility* of the imposed assumptions is an important quantity to report in empirical applications, since it can convey the *upper bound* of the credibility (most optimistic belief) for the imposed assumptions after observing data.[8] In fact, the posterior *plausibility* of the imposed assumptions is not specific in the current robust Bayes proposal, but it can be in principle computed as a by-product of the MCMC algorithm

[8] An alternative quantity that is informative for plausibility of the imposed assumptions is the prior-posterior odds of the nonemptiness of the identified set,

$$O_{F,S} = \frac{\tilde{\pi}_{\phi|Y}(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\})}{\tilde{\pi}_\phi(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\})}.$$

$O_{F,S}$ exceeding one indicates that the data are in favor of plausibility of the imposed assumptions.

in the Bayesian structural VAR analysis with the sign restricted impulse responses, although it has been rarely reported in the literature.

In the frequentist approach Moon, Shorfheide, and Granziera (2013), it is not straightforward to separate out inferential statement for the plausibility of assumptions from the confidence statement for the impulse response identified set. Observing narrow frequentist confidence intervals can be a consequence of sample information for the misspecification of the assumptions rather than precise sample information for the underlying small identified set, or *vice versa*. In contrast, the robust Bayes approach proposed above enables us to separately quantify these information on the basis of the posterior distribution of the reduced form parameters, by reporting both the posterior probability of having an empty identified set and the posterior bounds conditional on ϕ yielding nonempty identified set.

5.4 Asymptotic Property of the Posterior Bounds

This section concerns asymptotic consistency property of the posterior bounds proposed above. Let $\phi_0 \in \Phi$ be the value of the reduced form parameters that corresponds to the true sampling process of the data, and let $Y^T = (y_1, \ldots, y_T)$ denote a sample of size T. We assume posterior consistency for the reduced parameters, meaning $\lim_{T \to \infty} \pi_{\phi|Y^T}(B) = 1$ for every B open neighborhood of ϕ_0 for almost every sampling sequence Y^T in terms of the true sampling distribution of Y^T. The posterior consistency of the Gaussian reduced form VAR is standard, and we let the posterior consistency for ϕ be given in the following analysis.

If the identified set of r, $IS_r(\phi|F,S)$, viewed as a set-valued map of ϕ is non-empty and a continuous correspondence at $\phi = \phi_0$, then the posterior mean bounds constructed in Proposition 5.1 (ii) is consistent to the convex hull of $IS_r(\phi_0|F,S)$, as claimed in the next proposition.

Proposition 5.2 *Suppose that $IS_r(\phi|F,S)$ is a non-empty and continuous correspondence at $\phi = \phi_0$,*

(i) $\lim_{T\to\infty} \pi_{\phi|Y^T}(\{\phi : d_H(IS_r(\phi|F,S), IS_r(\phi_0|F,S)) > \epsilon\}) = 0$ *for almost every sampling sequence Y^T, where $d_H(\cdot,\cdot)$ is the Hausdorff distance.*

(ii) Let $[l(\phi_0), u(\phi_0)]$ be the convex hull of $IS_r(\phi_0|F,S)$. The range of the posterior means converges to $[l(\phi_0), u(\phi_0)]$,

$$\int_\Phi l(\phi)\, d\pi_{\phi|Y^T} \to l(\phi_0) \quad \text{and}$$
$$\int_\Phi u(\phi)\, d\pi_{\phi|Y^T} \to u(\phi_0), \quad \text{as } T \to \infty,$$

for almost every sampling sequence Y^T.

Proof. See Appendix A. ■

The first claim of this proposition shows that the identified set $IS_r(\phi|F,S)$ viewed as a random set induced by the posterior of ϕ converges to the true identified set in the Hausdorff metric. If $IS_r(\phi_0|F,S)$ is convex, as can be implied by the assumptions of Lemma 4.1 or 4.2, the posterior consistency in the sense of Proposition 5.2 (i) implies that at every credibility level $0 < \alpha < 1$ the robustified credible region constructed in (Step 5) of Algorithm 5.1 converges to the true identified set. On the other hand, if the true identified set is non-convex, then, the robustified credible region constructed by Algorithm 5.1 yields an interval estimate that is consistent to only a convex hull of the true identified set. The second statement in this proposition claims that the range of posterior means is also consistent to the convex hull of the true identified set. That is, if $IS_r(\phi_0|F, S)$ is known to be convex, the range of posterior means is consistent to the true identified set.

The continuity of $IS_r(\phi|F, S)$ at $\phi = \phi_0$ assumed in this proposition is crucial for guaranteeing consistency of the robust Bayes procedure. The continuity of $IS_r(\phi|F, S)$ can be obtained from a set of more primitive conditions involving rank conditions for the coefficient matrices of the zero and sign restrictions in the neighborhood of ϕ_0. Under the setup as considered in Lemma 4.1 and Lemma 4.2, Appendix B characterizes a sufficient condition under which the impulse response identified set becomes a continuous correspondence.

6 An Empirical Example

We illustrate a use of the posterior bound analysis developed above in a four-variable SVAR, where the vector of observables consists of a nominal interest rate i_t, real GDP y_t, inflation rate π_t, and real money balances m_t. The data set we use is from Aruoba and Schorfheide (2011), and it is used in the empirical illustration of Moon et al (2013). The data are quarterly observations for the period from 1965:I to 2005:I, constructed from the FRED2 database of Federal Reserve Bank of St. Louis. For the details of the construction of the variables, see Aruoba and Schorfheide (2011).

We specify the four variable structural VAR as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} i_t \\ \Delta y_t \\ \pi_t \\ m_t \end{pmatrix} = c + \sum_{l=1}^{2} A_l \begin{pmatrix} i_{t-l} \\ \Delta y_{t-l} \\ \pi_{t-l} \\ m_{t-l} \end{pmatrix} + \begin{pmatrix} \epsilon_{i,t} \\ \epsilon_{y,t} \\ \epsilon_{\pi,t} \\ \epsilon_{m,t} \end{pmatrix},$$

where we specify the lag to be $p = 2$. The output variable is transformed to the first difference (GDP growth), since without first-differencing y_t, the implied reduced form VAR is non-invertible with a high posterior probability. The order of variables presented here are determined in such way that the set of zero restrictions introduced below are compatible with $f_1 \geq f_2 \geq f_3 \geq f_4$. Suppose that the impulse response function of interest is the output response to a monetary policy shock, $\frac{\partial y_{t+h}}{\partial \epsilon_{i,t}}$, i.e., $\tilde{\imath} = 2$ and $j = 1$ in our notation of the previous sections. For the sign normalizations, we restrict the diagonal elements of the coefficient matrix in the left-hand side to be nonnegative, so the output response is estimated with respect to a unit standard deviation contractionary monetary policy shock.

We consider several specifications of the SVARs, where each specification uses a different combination of the following set of zero restrictions and sign restrictions.

Restrictions:

(i) The monetary authority does not respond to the contemporaneous GDP growth, $a_{12} = 0$.

(ii) The instantaneous impulse response of the real GDP to a monetary policy shock is zero, $IR^0(\Delta y, i) = 0$.

(iii) The long-run impulse response of the real GDP level to a monetary policy shock is zero, $CIR^\infty(\Delta y, i) \approx \sum_{h=1}^{H} \frac{\partial \Delta y_{t+h}}{\delta \epsilon_{i,t}} = 0$, with $H = 80$.

(iv) The inflation response to a contractionary monetary policy shock is nonpositive for one quarterly period, $\frac{\partial \pi_{t+h}}{\partial \epsilon_{i,t}} \leq 0$ for $h = 0, 1$, the interest rate response is nonnegative for one quarterly period, $\frac{\partial i_{t+h}}{\partial \epsilon_{i,t}} \geq 0$ for $h = 0, 1$, and the responses of the real money balances is nonpositive for one quarterly period, $\frac{\partial m_{t+h}}{\partial \epsilon_{i,t}} \geq 0$, for $h = 0, 1$.

To compare identifying power and informativeness of each restriction, we conduct the posterior bound analysis for seven different combinations of the restrictions. Table 1 summarizes the combinations of the restrictions we consider. The restrictions (i) through (iii) are zero restrictions that constrain the first column vector of Q, so $f_1 = 1$ if only one assumption out of (i) - (iii) are imposed (Model II and IV), and $f_1 = 2$ if two out of (i) - (iii) are imposed. On the other hand, no zero restrictions are placed on the other columns of Q, so $f_2 = f_3 = f_4 = 0$ hold for all the models. The sign restrictions on the impulse responses are given by (iv), which are identical to those considered in Moon, Shorfheide, and Granziera (2013). We impose the sign restrictions on all the specifications. Note that, in the all the models considered, the output impulse response functions are set-identified, so the set of priors defined above will yield the set of posteriors, and the posterior bounds are not supposed to collapse to a point. Furthermore, the current setup satisfies the conditions of Lemma 4.1 and 4.2, so that the identified set of the impulse responses are guaranteed to be convex for every ϕ. Consequently, the posterior bounds to be reported are consistent with the true identified set if additional regularity assumptions listed in Appendix B holds at the true ϕ.

Table 1: Definition of Models and Posterior Plausibility

Restriction \ Model	I	II	III	IV	V	VI	VII
(i) $a_{12} = 0$	-	O	-	-	O	O	-
(ii) $IR^0(y, i) = 0$	-	-	O	-	O	-	O
(iii) $IR^\infty(y, i) = 0$	-	-	-	O	-	O	O
(iv) sign restrictions	O	O	O	O	O	O	O
$\Pr\left(IS_r(\phi \mid F, S) \neq \emptyset \mid data\right)$	1.00	1.00	1.00	1.00	0.99	0.93	0.98

Note: "O" indicates the restriction is imposed

We set a prior for reduced form parameters ($\tilde{\pi}_\phi$ defined in Section 4) common to all the models and it is specified to be improper $d\tilde{\pi}_\phi(B, \Sigma) \propto |\Sigma|^{-\frac{4+1}{2}}$. This prior for ϕ corresponds to the

Jeffreys' prior for the reduced form Gaussian VAR, and the posterior for ϕ is nearly identical to the likelihood with the current sample size. The bottom row of Table 1 presents the posterior probabilities for plausibility of the imposed restrictions (nonemptiness of the identified set). In all the specifications considered, these probabilities are approximated to be either one or nearly one for the current sample.

In addition to the posterior bound analysis, we draw a single prior Bayes inference for the purpose of assessing how much extra information is added to the posterior inference by the non-updated part of the prior. We introduce a prior for Q that is similar to the agnostic Bayesian inference procedure of Uhlig (2005). Specifically, we obtain the approximated posterior for the impulse responses based on the MCMC draws of the impulse responses. The draws for the impulse responses are obtained by iterating Step (2.1) - (2.3) of Algorithm 4.1, and retaining the draws of Q satisfying the sign restrictions.

Figure 3 and 4 present the posterior distributions and the posterior bounds for impulse responses. In implementing Algorithm 4.1, we draw ϕ's until we obtain 1000 realizations of nonempty identified set $IS(\phi|S, F)$. In all the models considered, we employ the non-linear optimization step of Algorithm 4.1 (Step 3). Figure 3 and 4 summarize the marginal distribution of the impulse response at each horizon, and so the figure does not summarize the dependence of the responses across different horizons. Since the same prior is used for ϕ in every model and the posterior probabilities of having nonempty identified sets are close to one for all the models, the posterior bounds differ across the models only through the difference of the imposed zero and/or sign restrictions. Model I in Figure 3 shows that the posterior of ϕ combined with only the sign restrictions do not lead to informative inference for output responses; their posterior distributions vary over a wide range depending on a prior for Q, as we can observe that the posterior mean bounds are as wide as the posterior credible region of the single prior Bayesian procedure. Note that the posterior mean bounds and the robustified credible regions are as wide as the point estimator for the identified sets and the frequentist confidence intervals for them reported in Moon, Shorfheide, and Granziera (2013). These similarities to the frequentist inference for impulse response identified sets is not surprising given the consistency property of the posterior bounds (Proposition 5.1). When one zero restriction is additionally imposed (Model II - IV), the posterior mean bounds and the robustified credible regions get substantially tighter, although an informative conclusion is hard to draw (except the negative impact in the short horizon in Model III). With two zero restrictions additionally imposed (Model V - VII), the posterior mean bounds become informative for the sign of the output response for short to middle-range horizons. Specifically, when the imposed zero restrictions include restriction (ii) (Model V and VII), the range of posterior means of output responses are negative for $h = 0$ up to about $h = 10$. On the other hand, if restriction (i) and (iii) are jointly imposed, the range of posterior means is positive for short horizons, and we obtain the opposite conclusions to Model V and VII. These results on relatively more informative posterior bounds show that, despite the lack of point-identification, the posterior is less sensitive to a choice of prior for Q once any of the two zero restrictions are imposed.

It is worth noting that both the posterior mean bounds and the lower credible region become tighter as more restrictions are added. As far as the posterior probabilities of nonemptiness of the identified set are one, this monotonic gain in informativeness of the posterior bounds holds *irrespective of the realized values of the observations*, since adding zero equality or sign restrictions monotonically reduce the size of the prior class without changing the posterior of ϕ. This property of "more restrictions, more informative inference" does not necessarily hold when we report frequentist confidence intervals for the true identified set.

7 Concluding Remarks

This paper develops a robust Bayes inference for non-identified structural vector autoregressions. The proposed procedure reports the range of posterior probabilities and posterior means for impulse responses, when a prior varies over the class that consists of any priors for the non-identified components in the model. The range of posterior quantities induced by such class of priors can be interpreted as a Bayesian inference for the identified set, and the posterior mean bounds and the robustified credible region converge to the true identified set. The posterior bounds are easy to compute, even when the sign restrictions are placed for multiple shocks, and it can be applied to a large class of non-identified SVARs involving zero restrictions and sign restrictions. We consider that the offered procedure is a useful tool to separate the information for the impulse responses based on the sample likelihood from any prior input that is not updated by data. With a diffuse prior of the reduced form parameters, the posterior bound analysis also provides a way to summarize the shape of the observed likelihood by dealing with the flat region of likelihood by profiling rather than integration. Given that the shape of the likelihood is an important object to know for both Bayesians and frequentists, we believe both Bayesians and frequentists find the proposed robust Bayes analysis useful in summarizing and visualizing the information of the impulse response of interest contained in data.

Appendix

A Omitted Proofs

The proofs of Lemma 4.1 and 4.2 given below use the following notations. For given $\phi \in \Phi$, and $i = 1, \ldots, n$, let $\tilde{f}_i(\phi) \equiv rank(F_i(\phi))$. Let $\mathcal{V}_i(\phi)$ be the linear subspace of $\mathcal{R}^n$ that is orthogonal to the row vectors of $F_i(\phi)$. If no zero restrictions are placed for q_i, we interpret $\mathcal{V}_i(\phi)$ to be $\mathcal{R}^n$. Note that, the dimension of $\mathcal{V}_i(\phi)$ is equal to $n - \tilde{f}_i(\phi)$. We let $\mathcal{H}_i(\phi)$ be the half-space in $\mathcal{R}^n$ defined by $\left\{z \in \mathcal{R}^n : \sigma^i(\phi)' z \geq 0\right\}$, where $\sigma^i(\phi)$ is the i-th column vector of Σ_{tr}^{-1}. The unit sphere in $\mathcal{R}^n$ is denoted by $\mathcal{S}^n$. Given linearly independent vectors, $A = [a_1, \ldots, a_l] \in \mathcal{R}^{n \times l}$, denote

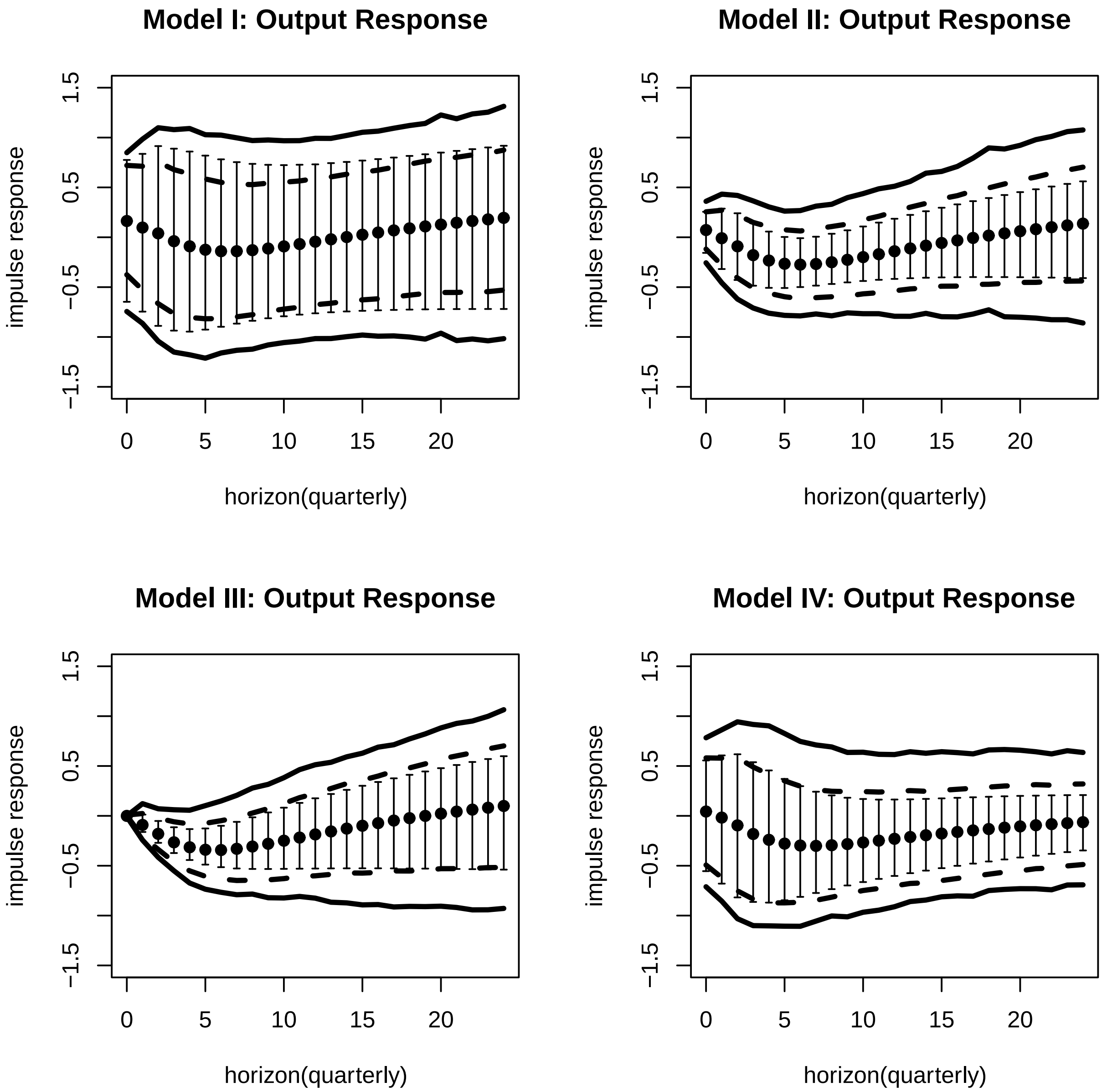

Figure 3: **Plots of Output Impulse Responses.** See Table 1 for the definition of each model. In each figure, the points plot the posterior means, the vertical bars show the posterior mean bounds, the dashed curves connect the upper/lower bounds of the highest posterior density regions with credibility 90%, and the solid curves connect the upper/lower bounds of the posterior lower credible regions with credibility 90%. The posterior mean and the highest posterior density regions are computed from the single prior Bayesian procedure described in the main text.

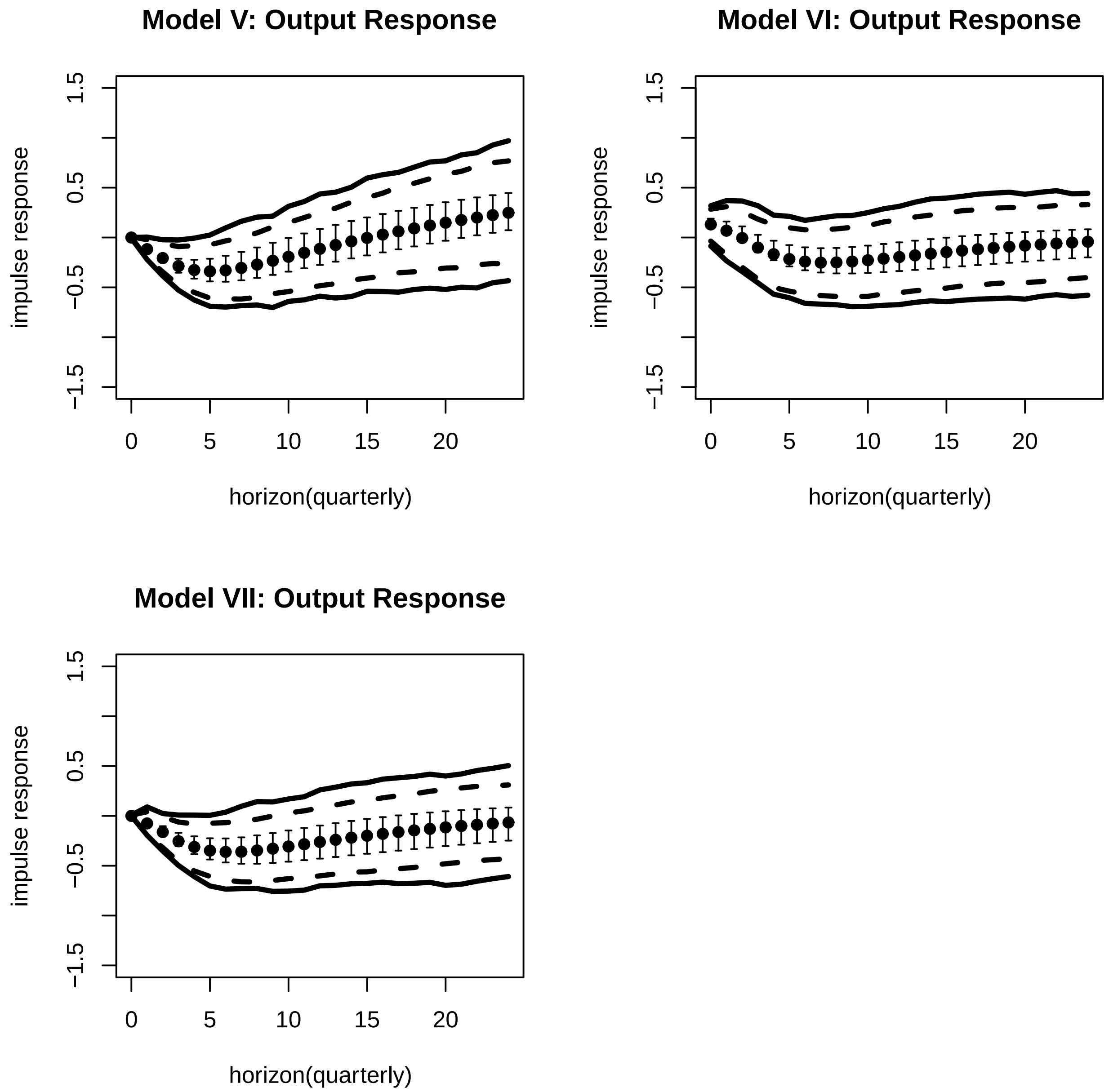

Figure 4: **Plots of Output Impulse Responses for Model IV - VII.** See Table 1 for the definition of each model. See the caption of Figure 1 for remarks.

the linear subspace in $\mathcal{R}^n$ that is orthogonal to the column vectors of A by $\mathcal{P}(A)$. Note that the dimension of $\mathcal{P}(A)$ is $n-l$.

Proof of Lemma 4.1 (i). Fix $\phi \in \Phi$, and denote $Q_{1:i} = [q_1, \ldots, q_i]$ be an $n \times i$ matrix of orthogonal vectors in $\mathcal{R}^n$. The set of feasible Q's satisfying the zero restrictions, $\mathcal{Q}(\phi|F)$, can be written in the following recursive manner,

$$\begin{aligned}
Q &= [q_1, \ldots, q_n] \in \mathcal{Q}(\phi|F) \\
&\quad \text{if and only if } Q = [q_1, \ldots, q_n] \text{ satisfies} \\
q_1 &\in D_1(\phi) \equiv \mathcal{V}_1(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^n, \\
q_2 &\in D_2(\phi, q_1) \equiv \mathcal{V}_2(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1) \cap \mathcal{S}^n, \\
q_3 &\in D_3(\phi, Q_{1:2}) \equiv \mathcal{V}_3(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}) \cap \mathcal{S}^n, \\
&\vdots \\
q_j &\in D_j\left(\phi, Q_{1:(j-1)}\right) \equiv \mathcal{V}_j(\phi) \cap \mathcal{H}_j(\phi) \cap \mathcal{P}(Q_{1:(j-1)}) \cap \mathcal{S}^n, \\
&\vdots \\
q_n &\in D_n\left(\phi, Q_{1:(n-1)}\right) \equiv \mathcal{V}_n(\phi) \cap \mathcal{H}_n(\phi) \cap \mathcal{P}(Q_{1:(n-1)}) \cap \mathcal{S}^n.
\end{aligned} \tag{A.1}$$

where $D_i\left(\phi, Q_{1:(i-1)}\right) \subset \mathcal{R}^n$ is the set of feasible q_i's given $Q_{1:(i-1)} = [q_1, \ldots, q_{i-1}]$, $(i-1)$ orthonormal vectors in $\mathcal{R}^n$ preceding i. First, nonemptiness of the identified set for $r = c_{\tilde{\imath}h}(\phi)\, q_j$ follows if the feasible domains of the orthogonal vector, $D_i\left(\phi, Q_{1:(i-1)}\right)$ is nonempty at every $i = 1, \ldots, n$. Note that, by the assumption of $f_1 \leq n-1$, $\mathcal{V}_1(\phi) \cap \mathcal{H}_1(\phi)$ is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension $n - \tilde{f}_1(\phi) \geq n - f_1 \geq 1$. Hence, $D_1(\phi)$ is nonempty for every $\phi \in \Phi$. For $i = 2, \ldots, n$, $\mathcal{V}_i(\phi) \cap \mathcal{H}_i(\phi) \cap \mathcal{P}(Q_{1:(i-1)})$, is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension at least

$$\begin{aligned}
n - \tilde{f}_i(\phi) - \dim(\mathcal{P}(Q_{1:(n-1)})) &\geq n - f_i - (i-1) \\
&\geq 1,
\end{aligned}$$

where the last inequality follows by the assumption of $f_i \leq n - i$. Hence, $D_i\left(\phi, Q_{1:(i-1)}\right)$ is nonempty for every $\phi \in \Phi$. Thus, we conclude that $\mathcal{Q}(\phi|F)$ is nonempty, and this implies nonemptiness of the impulse response identified sets for every $\tilde{\imath} \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$.

Next, we show convexity of $IS_r(\phi|F)$. We first consider the case of $j = 1$, and subsequently generalize the claim to the cases for $j > 1$ by mathematical induction. Since $f_1 \leq n-1$ by assumption, the feasible domain for q_1, $D_1(\phi)$, is the half-space of a linear subspace in $\mathcal{R}^n$ with dimension $n - \tilde{f}_1(\phi) \geq n - f_1 \geq 1$, intersected with the unit sphere, so $D_1(\phi)$ is a compact and path-connected subset in $\mathcal{R}^n$. (In case of $f_1 = n-1$ and $rank(F_1(\phi)) = n-1$, $D_1(\phi)$ is a singleton.) Since the range of a continuous function on a path-connected domain is convex, the convexity of the identified set of $r = c'_{\tilde{\imath}h}(\phi) q_1$ follows.

We now generalize the claim to $j \geq 2$. Let us view the feasible domain of q_j, $D_j\left(\phi, Q_{1:(j-1)}\right)$, given in (A.1) as a set-valued map from $Q_{1:(j-1)} \subset \mathcal{R}^{n \times (j-1)}$ to $\mathcal{R}^n$. We denote the domain of this set-valued map by

$$\mathcal{Q}_{1:(j-1)}(\phi|F) = \left\{Q_{1:(j-1)} : q_1 \in D_1(\phi), q_2 \in D_2(\phi, q_1), \ldots, q_{j-1} \in D_{j-1}\left(\phi, Q_{1:(j-2)}\right)\right\}. \tag{A.2}$$

We first present an auxiliary lemma, whose proof is postponed to a later part of this section.

Lemma A1: Fix $\phi \in \phi$. *Assume $\mathcal{Q}_{1:(j-1)}(\phi|F)$ is a compact and path-connected subset in $R^{n\times(j-1)}$. Then, the graph of the set-valued map $D_j\left(\phi, Q_{1:(j-1)}\right)$,*

$$\mathcal{Q}_{1:j}(\phi|F) \equiv \left\{Q_{1:j} = \left[Q_{1:(j-1)}, q_j\right] \in \mathcal{R}^{n\times j} : q_j \in D_j\left(\phi, Q_{1:(j-1)}\right), Q_{1:(j-1)} \in \mathcal{Q}_{1:(j-1)}(\phi|F)\right\} \tag{A.3}$$

is a compact and path-connected set in $\mathcal{R}^{n\times j}$, and its projection onto the range space for q_j,

$$\mathcal{E}_j(\phi) \equiv \bigcup_{Q_{1:(j-1)}\in\mathcal{Q}_{1:(j-1)}(\phi|F)} D_j\left(\phi, Q_{1:(j-1)}\right), \tag{A.4}$$

is a compact and path-connected set in $\mathcal{R}^n$.

Note that for the case of $j = 2$, $\mathcal{Q}_{1:(j-1)}(\phi|F)$ is given by $D_1(\phi)$. We have already shown that $D_1(\phi)$ is a compact and path-connected set in $\mathcal{R}^n$. So, successive applications of Lemma A.1 shows $\mathcal{E}_j(\phi) = \bigcup_{Q_{1:(j-1)}\in\mathcal{Q}_{1:(j-1)}(\phi|F)} D_j\left(\phi, Q_{1:(j-1)}\right)$ is a compact and path-connected set in $\mathcal{R}^n$. Since the identified set of $r = c'_{ih}(\phi)q_j$ is given by the range of a continuous function of q_j, the compactness and path-connectedness of $\mathcal{E}_j(\phi)$ implies compactness and convexity of the identified set of r. ■

Proof of Lemma 4.1 (ii). Let $[q_1, \ldots, q_j]$ be orthonormal vectors obtained by the algorithm given in the statement of the lemma. Under the given rank conditions, $[q_1, \ldots, q_j]$ are unique up to combinations of the signs for each column vector. If we focus on q_j, the sign normalization restriction $q_j'\sigma^j \geq 0$ allows us to uniquely pin it down if $q_j'\sigma^j \neq 0$, as assumed, holds. The uniqueness of q_j implies that the identified for r is a singleton. Hence, the conclusion follows. ■

Next we provide a proof of Lemma A.1 appeared in the proof of Lemma 4.1 (i).

Proof of Lemma A.1. To show that $\mathcal{Q}_{1:j}(\phi|F)$ is a compact given $\mathcal{Q}_{1:(j-1)}(\phi|F)$ is compact, it suffices to show that $D_j\left(\phi, Q_{1:(j-1)}\right)$ is a compact-valued correspondence. Recall that $D_j\left(\phi, Q_{1:(j-1)}\right)$ is the half-space of the linear subspace with dimension greater than or equal to 1 intersected with unit sphere (see (A.1)). Hence, $D_j\left(\phi, Q_{1:(j-1)}\right)$ is a compact and path-connected set in $\mathcal{R}^n$ at every $Q_{1:(j-1)} \in \mathcal{Q}_{1:(j-1)}(\phi|F)$. Furthermore, set-valued map $D_j\left(\phi, Q_{1:(j-1)}\right) : \mathcal{Q}_{1:(j-1)}(\phi|F) \rightrightarrows \mathcal{R}^n$ is a bounded and upper-semicontinuous correspondence[9] defined on a compact domain, so the graph of $D_j\left(\phi, Q_{1:(j-1)}\right)$, $\mathcal{Q}_{1:j}(\phi|F)$, is closed and bounded (see, e.g., Proposition 1.4.8 of Aubin and Frankowska (1990)).

[9]Upper-semicontinuity of $D_j\left(\phi, Q_{1:(j-1)}\right)$ in $Q_{1:(j-1)}$ holds by the following argument. Consider a sequence $Q^v_{1:(j-1)}$ in $\mathcal{Q}_{1:(j-1)}(\phi|F)$ with limit $Q_{1:(j-1)}$, and let $q^v_j \in D_j\left(\phi, Q^v_{1:(j-1)}\right)$. Since q^v_j's lie on a compact set, q^v_j has a

To show path-connectedness of $\mathcal{Q}_{1:j}(\phi|F)$, we show that there exists a continuous path between any two points, $\left(q_j, Q_{1:(j-1)}\right)$ and $\left(\tilde{q}_j, \tilde{Q}_{1:(j-1)}\right)$, in $\mathcal{Q}_{1:j}(\phi|F)$. Since $D_j\left(\phi, Q_{1:(j-1)}\right)$: $\mathcal{Q}_{1:(j-1)}(\phi|F) \rightrightarrows \mathcal{R}^n$ is a nonempty upper-semicontinuous correspondence defined on a path-connected domain, there exists a continuous function $g : \mathcal{Q}_{1:(j-1)}(\phi|F) \to \mathcal{R}^n$ such that $g(Q_{1:(j-1)}) \in D_j\left(\phi, Q_{1:(j-1)}\right)$ holds for every $Q_{1:(j-1)} \in \mathcal{Q}_{1:(j-1)}(\phi|F)$. Consider constructing a continuous path from $\left(q_j, Q_{1:(j-1)}\right)$ to $\left(\tilde{q}_j, \tilde{Q}_{1:(j-1)}\right)$, by connecting (i) $\left(q_j, Q_{1:(j-1)}\right)$ to $(g(Q_{1:(j-1)}), Q_{1:(j-1)})$, (ii) $(g(Q_{1:(j-1)}), Q_{1:(j-1)})$ to $(g(\tilde{Q}_{1:(j-1)}), \tilde{Q}_{1:(j-1)})$, and (iii) $(g(\tilde{Q}_{1:(j-1)}), \tilde{Q}_{1:(j-1)})$ to $(\tilde{q}_j, \tilde{Q}_{1:(j-1)})$. Existence of continuous paths for (i) and (iii) follow by the path-connectedness of $D_j\left(\phi, Q_{1:(j-1)}\right)$ and $D_j\left(\phi, \tilde{Q}_{1:(j-1)}\right)$, respectively. Existence of a continuous path for (ii) follows since g is continuous and the domain $\mathcal{Q}_{1:(j-1)}(\phi|F)$ is path-connected. Thus, we obtain path-connectedness of $\mathcal{Q}_{1:j}(\phi|F)$.

A projection of a compact and path-connected set $\mathcal{Q}_{1:j}(\phi|F)$ onto a lower dimensional space is a continuous map, and it preserves compactness and path-connectedness, so $\mathcal{E}_j(\phi)$ is compact and path-connected. ■

Proof of Lemma 4.2. Under Condition 1 of this lemma, Lemma 4.1 (ii) implies that $IS_r(\phi|F, S)$ is either a singleton or an empty set, depending on whether sign restrictions $S_i(\phi)\, q_j \geq 0$ holds or not at the a.s. uniquely determined q_j given ϕ. Hence, $IS_r(\phi|F, S)$ is convex whenever it is nonempty.

Under Condition 3 of this lemma, the algorithm of Lemma 4.1 (ii) combined with the sign normalization restrictions pins down uniquely the first i^* column vectors of $Q \in \mathcal{Q}(\phi|F, S)$, if $\mathcal{Q}(\phi|F, S)$ is non-empty. We denote them by $[q_1^*, \ldots, q_{i^*}^*]$. Let the orthogonal matrices that attain the lower bound and the upper bound of $\{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F, S)\}$ be $Q = [q_1^*, \ldots, q_{i^*}^*, q_{i^*+1}, \ldots, q_n]$ and $\tilde{Q} = [q_1^*, \ldots, q_{i^*}^*, \tilde{q}_{i^*+1}, \ldots, \tilde{q}_n]$, respectively. If $j > i^* + 1$, the j-th column vectors of Q and $\tilde{Q}$

converging subsequence with a limit denoted by q_j. Since $\begin{pmatrix} F_j(\phi) \\ Q_{1:(j-1)}^v \end{pmatrix} q_j^v = 0$ and $\sigma^j(\phi)' q_j^v \geq 0$ hold for all v, and by continuity of inner product, $\begin{pmatrix} F_j(\phi) \\ Q_{1:(j-1)} \end{pmatrix} q_j = 0$ and $\sigma^j(\phi)' q_j \geq 0$ hold, implying $q_j \in D_j\left(\phi, Q_{1:(j-1)}\right)$. Hence, $D_j\left(\phi, Q_{1:(j-1)}\right)$ is upper-semicontinuous in $Q_{1:(j-1)}$.

satisfy

$$\begin{pmatrix} F_j(\phi) \\ q_1^{*\prime} \\ \vdots \\ q_{i^*}^{*\prime} \\ q_{i^*+1}^{\prime} \\ \vdots \\ q_{j-1}^{\prime} \end{pmatrix} q_j = 0, \quad S(\phi, Q) \geq 0, \quad \text{and}$$

$$\begin{pmatrix} F_j(\phi) \\ q_1^{*\prime} \\ \vdots \\ q_{i^*}^{*\prime} \\ \tilde{q}_{i^*+1}^{\prime} \\ \vdots \\ \tilde{q}_{j-1}^{\prime} \end{pmatrix} \tilde{q}_j = 0, \quad S(\phi, \tilde{Q}) \geq 0.$$

Define $q_j(\lambda) = \frac{\lambda q_j + (1-\lambda)\tilde{q}_j}{\|\lambda q_j + (1-\lambda)\tilde{q}_j\|}$, for $\lambda \in [0,1]$. Note that $\|\lambda q_j + (1-\lambda)\tilde{q}_j\| \neq 0$, since $\|\lambda q_j + (1-\lambda)\tilde{q}_j\| = 0$ happens only when $q_j = -\tilde{q}_j$, but this possibility is excluded ϕ-a.s. from the sign normalization restriction. We now show that, for any $\lambda \in [0,1]$, there exists $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ whose j-th column vector is $q_j(\lambda)$. Consider constructing the orthogonal vectors, $q_i(\lambda)$, $i = i^* + 1, \ldots, (j-1)$ recursively by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1^{*\prime} \\ \vdots \\ q_{i^*}^{*\prime} \\ q_{i^*+1}^{\prime}(\lambda) \\ \vdots \\ q_{i-1}^{\prime}(\lambda) \\ q_j(\lambda) \end{pmatrix} q_i(\lambda) = 0. \tag{A.5}$$

Note that the there exist $q_i(\lambda)$ solving these orthogonality conditions because under the assumption that $f_i \leq n - i - 1$, $i = i^* + 1, \ldots, j-1$, the matrices multiplied to $q_i(\lambda)$ have rank at most $n-1$. As for the orthogonal vectors for the columns $i = j+1, \ldots, n$, the under-identified situation $f_i \leq n - i$ guarantees existence of $[q_{j+1}(\lambda), \ldots, q_n(\lambda)]$, as successive applications of the orthogonal condition (A.5) for $i = j+1, \ldots, n$ can yield orthogonal vectors $[q_{j+1}(\lambda), \ldots, q_n(\lambda)]$. Thus, we obtain $Q(\lambda) = [q_1^*, \ldots, q_{i^*}^*, q_{i^*+1}(\lambda), \ldots, q_n(\lambda)]$, which belongs to $\mathcal{Q}(\phi|F,S)$ since $F(\phi, Q(\lambda)) = 0$ and $S_j(\phi) q_j(\lambda) \geq 0$ holds by the construction. Since $q_j(\lambda)$, $\lambda \in [0,1]$, forms a continuous path between q_j and $\tilde{q}_j$ and $r = c_{\tilde{\imath},h}(\phi) q_j$ is continuous in q_j, the identified set of $r = c_{\tilde{\imath},h}(\phi) q_j$ is given by a convex interval $[c_{\tilde{\imath},h}(\phi) q_j, c_{\tilde{\imath},h}(\phi) \tilde{q}_j]$. Hence, we conclude that $IS_r(\phi|F)$ is a connected interval.

If $j = i^* + 1$, then the j-th column vector of Q giving the lower and upper bounds of the impulse response satisfy

$$\begin{aligned} \begin{pmatrix} F_j(\phi) \\ q_1^{*\prime} \\ \vdots \\ q_{i^*}^{*\prime} \end{pmatrix} q_j &= 0, \quad S(\phi, Q) \geq 0, \quad \text{and} \\ \begin{pmatrix} F_j(\phi) \\ q_1^{*\prime} \\ \vdots \\ q_{i^*}^{*\prime} \end{pmatrix} \tilde{q}_j &= 0, \quad S(\phi, \tilde{Q}) \geq 0. \end{aligned}$$

Construct $q_j(\lambda)$ as above. Clearly, $q_j(\lambda)$ meets the orthogonality conditions and the sign restrictions, and it provides a continuous path between q_j and $\tilde{q}_j$.

Under Condition 2 of the lemma, we simply drop $[q_1^*, \ldots, q_{i^*}^*]$ from the preceding proof for the case of Condition 3, and the conclusion follows by adopting the same way of constructing a continuous path, $q_j(\lambda)$, $\lambda \in [0,1]$. ■

Proof of Proposition 5.1 (ii). The proof proceeds by applying the proof of Proposition 4.1 of Kitagawa (2012). Let $r(\phi, Q) = c_{ih}'(\phi) q_j$ be the impulse response of interest. By Lemma A.4 of Kitagawa (2012) and Proposition 10.3 of Denneberg (1994), the upper bounds of the posterior mean of $r(\phi, Q)$ satisfies the following equality,

$$\sup_{\pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}} \int r(\phi, Q) d\pi_{\phi Q|Y} = \int r(\phi, Q) d\pi^*_{\phi Q|Y},$$

where the integral with respect to the upper probability $\int r(\phi, Q) d\pi^*_{\phi Q|Y}$ stands for the generalized Choquet integral (Denneberg (1994), pp62),

$$\int r(\phi, Q) d\pi^*_{\phi Q|Y} = \int_{-\infty}^{0} \left[\pi^*_{\phi Q|Y}(\{r(\phi, Q) \geq \tilde{r}\}) - 1\right] d\tilde{r} + \int_0^\infty \pi^*_{\phi Q|Y}(\{r(\phi, Q) \geq \tilde{r}\}) d\tilde{r}.$$

By Proposition 5.1 (i), it holds

$$\begin{aligned} \pi^*_{\phi Q|Y}(\{r(\phi, Q) \geq r\}) &= \pi^*_{r|Y}(\{r \geq \tilde{r}\}) \\ &= \pi_{\phi|Y}(IS(\phi|F, S) \cap \{r \geq \tilde{r}\} \neq \emptyset). \end{aligned}$$

Note that $IS(\phi|F, S) \cap \{r \geq \tilde{r}\} \neq \emptyset$ is true if and only if $\{u(\phi) \geq \tilde{r}\}$. Hence, we have

$$\begin{aligned} \int r(\phi, Q) d\pi^*_{\phi Q|Y} &= \int_{-\infty}^{0} \left[\pi_{\phi|Y}(u(\phi) \geq \tilde{r}) - 1\right] d\tilde{r} + \int_0^\infty \pi_{\phi|Y}(u(\phi) \geq \tilde{r}) d\tilde{r} \\ &= -\int_{-\infty}^{0} \pi_{\phi|Y}(u(\phi) < \tilde{r}) d\tilde{r} + \int_0^\infty \pi_{\phi|Y}(u(\phi) \geq \tilde{r}) d\tilde{r} \\ &= E_{\phi|Y}(u(\phi)), \end{aligned}$$

where the last line follows by interchanging the order of integrations. The lower bound of the posterior means can be obtained similarly by replacing $r(\phi, Q)$ above with $-r(\phi, Q)$. Any posterior means between the lower and upper bounds can be obtained by a mixture of the priors attaining the lower and upper bounds, so the range of the posterior means is convex. ■

Proof of Proposition 5.2. (i) Let $\epsilon > 0$ be arbitrary, and denote the identified set of an impulse response by $IS(\phi)$ for short. By the assumption of continuity of the identified set correspondence, there exists an open neighborhood G of ϕ_0 such that $d_H\left(H\left(\phi\right), H\left(\phi_0\right)\right) < \epsilon$ holds for every $\phi \in G$. Consider

$$\begin{aligned} F_{\phi|Y^T}\left(\{\phi : d_H\left(IS(\phi), IS(\phi_0)\right) > \epsilon\}\right) &= F_{\phi|Y^T}\left(\{\phi : d_H\left(IS(\phi), IS(\phi_0)\right) > \epsilon\} \cap G\right) \\ &\quad + F_{\phi|Y^T}\left(\{\phi : d_H\left(IS(\phi), IS(\phi_0)\right) > \epsilon\} \cap G^c\right) \\ &\leq F_{\phi|Y^T}\left(G^c\right), \end{aligned}$$

where the last line follows because $\{\phi : d_H\left(IS(\phi), IS(\phi_0)\right) > \epsilon\} \cap G = \emptyset$ by the construction of G. The posterior consistency of ϕ yields $\lim_{T\to\infty} F_{\phi|Y^T}\left(G^c\right) = 0$, $p\left(Y^\infty|\phi_0\right)$-a.s., so $\lim_{T\to\infty} F_{\phi|Y^T}\left(\{\phi : d_H\left(IS(\phi), \right.\right.$ 0 holds $p\left(Y^\infty|\phi_0\right)$-a.s.

(ii) Continuity of the identified set correspondence implies that $l(\phi)$ and $u(\phi)$ is continuous at ϕ_0. Since the assumption of the posterior consistency of ϕ implies that $\pi_{\phi|Y^T}$ converges weakly to the Dirac measure at ϕ_0 for almost all Y^T. Hence, $\int_\Phi l(\phi) d\pi_{\phi|Y^T} \to l(\phi_0)$ and $\int_\Phi u(\phi) d\pi_{\phi|Y^T} \to u(\phi_0)$ as $T \to \infty$ holds for almost all Y^T. ■

B Continuity of Identified Set Correspondence

In this appendix, we provide a set of conditions, under which the identified set for impulse responses becomes a continuous correspondence. We treat the case with only the zero restriction and the case with both zero and sign restrictions separately.

B.1 Only Zero Restrictions

This section of the appendix concerns the case as considered in Lemma 4.1, i.e., the imposed restrictions compose of zero restrictions satisfying $f_i \leq n - i$ for all $i = 1, \ldots, n$. As for the set of feasible Q's, we follow the representation of $\mathcal{Q}(\phi|F)$ given in (A.1) and the notations defined in the proof of Lemma 4.1.

Let our object of interest be the impulse responses with respect to a j-th shock, and let ϕ_0 be the true value of reduced form parameters, at which continuity of the identified set correspondence is concerned. By Lemma 4.1, the impulse response identified set is compact and convex, so, in order to show continuity of the identified set correspondence, it suffices to show continuity of $l(\phi)$ and $u(\phi)$ at $\phi = \phi_0$. Note that $l(\phi)$ and $u(\phi)$ can be seen as minimized and maximized values of $\tilde{r}(\phi, Q_{1:j}) = c_{ih}'(\phi) q_j$ subject to $Q_{1:j} \in \mathcal{Q}_{1:j}(\phi|F)$, where the second argument of $\tilde{r}(\phi, Q_{1:j})$

emphasizes that we see $Q_{1:j}$ as the choice variable in the optimizations (the rest of column vectors $q_{j+1}, \ldots, q_n$ can be ignored since the zero restrictions placed for them do not constrain $Q_{1:j}$ due to the sequential structure of (A.1)). Since objective function $\tilde{r}(\phi, Q_{1:j})$ is continuous in ϕ and $Q_{1:j}$, the Maximum Theorem (see, e.g., Theorem 9.14 of Sundaram (1996)) says continuity of $l(\phi)$ and $u(\phi)$ follows from continuity of a constraint set correspondence $\mathcal{Q}_{1:j}(\phi|F)$. Therefore, it suffices to focus on characterizing a condition for continuity of $\mathcal{Q}_{1:j}(\phi|F)$.

Continuity of $\mathcal{Q}_{1:j}(\phi|F)$ can be obtained by combining the following two lemmas. Proofs of these lemmas are given in a later part of this section.

Lemma B.1 *Assume*

1. *there exists $G \subset \Phi$ an open neighborhood of ϕ_0 such that $rank(F_1(\phi)) = f_1$ for all $\phi \in G$, and*
2. *the first column vector of the true Σ_{tr}^{-1}, $\sigma^1(\phi_0)$, is linearly independent of the row vectors of $F_1(\phi_0)$.*

Then, $D_1(\phi) = \left\{q_1 \in \mathcal{S}^n : F_1(\phi)q_1 = \mathbf{0}, \sigma^1(\phi_0)'q_1 \geq 0\right\}$ is a continuous correspondence at $\phi = \phi_0$.

Lemma B.2 *Let $i \in \{2, \ldots, n\}$ be fixed. Suppose that $\mathcal{Q}_{1:(i-1)}(\phi|F)$ is a continuous correspondence at $\phi = \phi_0$. Assume the following two conditions.*

1. *Let $\mathcal{K}_i(\phi) = \left\{ rank \begin{pmatrix} F_i(\phi) \\ Q'_{1:(i-1)} \end{pmatrix} : Q_{1:(i-1)} \in \mathcal{Q}_{1:(i-1)}(\phi|F) \right\}$ be the set of ranks that $(f_i + i - 1) \times n$ matrix $\begin{pmatrix} F_i(\phi) \\ Q'_{1:(i-1)} \end{pmatrix}$ can take as $Q_{1:(i-1)}$ varies over $\mathcal{Q}_{1:(i-1)}(\phi|F)$. There exists $G \subset \Phi$ an open neighborhood of ϕ_0 such that $\mathcal{K}_i(\phi) = \mathcal{K}_i(\phi_0)$ holds for all $\phi \in G$.*
2. *The i-th column vector of the true Σ_{tr}^{-1}, $\sigma^i(\phi_0)$, is linearly independent of the row vectors of $\begin{pmatrix} F_i(\phi_0) \\ Q'_{1:(i-1)} \end{pmatrix}$ for all $Q_{1:(i-1)} \in \mathcal{Q}_{1:(i-1)}(\phi_0|F)$.*

Then, $\mathcal{Q}_{1:i}(\phi|F)$ is a continuous correspondence at $\phi = \phi_0$.

Lemma B.1 concerns continuity of the constraint set correspondence for q_1 only, and Lemma B.2 concerns continuity of the constraint set correspondence for $Q_{1:i}$ given continuity of the one of $Q_{1:(i-1)}$. If the impulse responses of interest are those with respect to a shock in the first variable $(j = 1)$, Lemma B.1 combined with the Maximum theorem yields continuity of $l(\phi)$ and $u(\phi)$ at $\phi = \phi_0$. Lemma B.2 is used if the impulse responses of interest are those with respect to a shock in variable $j \geq 2$. In case of $i = 2$, $\mathcal{Q}_{1:(i-1)}(\phi|F)$ is guaranteed to be a continuous correspondence by Lemma B.1, so if Assumptions 1 and 2 hold sequentially from $i = 2$ to $(j-1)$, then recursive applications of Lemma B.2 proves that $\mathcal{Q}_{1:j}(\phi|F)$ is a continuous correspondence at $\phi = \phi_0$. Thus, as a corollary of Lemmas B.1 and B.2, we obtain continuity of $l(\phi)$ and $u(\phi)$, as formally stated in the next proposition.

Proposition B.1 *Assume the zero restrictions satisfy $f_i \leq n-i$ for $i = 1, 2, \ldots, n$. If the impulse responses of interest are those with respect to a shock in the first variable ($j = 1$), $r = c'_{\tilde{i}h}(\phi)q_1$, then Assumptions 1 and 2 of Lemma B.1 are sufficient for continuity of the impulse response identified set for all $\tilde{i} = 1, \ldots, n$ and $h = 0, 1, 2, \ldots$. If the impulse responses of interest are those with respect to a shock in the j -th variable with $j \geq 2$, $r = c'_{\tilde{i}h}(\phi)q_j$, then the impulse response identified set is continuous for all $\tilde{i} = 1, \ldots, n$ and $h = 0, 1, 2, \ldots$ if Assumptions 1 and 2 of Lemma B.1 hold and Assumptions 1 and 2 of Lemma B.2 hold for all $i = 1, \ldots, j-1$.*

The conditions specified in Lemmas B.1 and B.2 consist of two types of rank conditions. Assumption 1 in these lemmas says that the possible dimensions of the linear subspace that q_i can lie does not vary in ϕ in a neighborhood of ϕ_0. Assumption 2 in these lemmas says that the vector of the sign normalization restriction for q_i is linearly independent of the coefficient matrix of the zero restrictions for q_i.

Proof of Lemma B.1. Note first that $D_1(\phi)$ is a closed and bounded correspondence, so upper-semicontinuity and lower-semicontinuity of $D_1(\phi)$ can be defined in terms of sequences (see, e.g., Propositions 21 of Border (2013)),

- $D_1(\phi)$ is upper-semicontinuous (usc) at $\phi = \phi_0$ if and only if, for any sequence $\phi^v \to \phi_0$ and any $q_1^v \in D_1(\phi^v)$, there is a subsequence of q_1^v with limit in $D_1(\phi_0)$.
- $D_1(\phi)$ is lower-semicontinuous (lsc) at $\phi = \phi_0$ if and only if, $\phi^v \to \phi_0$ and $q_1^0 \in D_1(\phi_0)$ imply that there is a sequence $q_1^v \in D_1(\phi^v)$ with $q_1^v \to q_1^0$.

In the proofs given below, we use the same index v to denote a subsequence, just to compress a notational burden.

Usc: Since q_1^v is a sequence on a unit-sphere, it has a convergent subsequence $q_1^v \to q_1$. Since $q_1^v \in D_1(\phi^v)$, $F_1(\phi^v)q_1^v = \mathbf{0}$ and $\sigma^1(\phi^v)'q_1^v \geq 0$ hold for all v, and they hold at the limit as well since $F_1(\phi)$ and $\sigma^1(\phi)$ are continuous. Hence, $q_1 \in D_1(\phi_0)$.

Lsc: The proof of lsc proceeds in a similar manner to the proof of Lemma 3 in Moon et al (2013). Let $q_1^0 \in D_1(\phi_0)$, and define $\mathbf{P}^0 = F_1(\phi_0)'\left[F_1(\phi_0)F_1(\phi_0)'\right]^{-1}F_1(\phi_0)$ be the projection matrix onto the space spanned by the row vector of $F_1(\phi_0)$. By Assumption 1, $\mathbf{P}^0$ and $\mathbf{P}^v = F_1(\phi^v)'\left[F_1(\phi^v)F_1(\phi^v)'\right]^{-1}F_1(\phi^v)$ are well-defined for all large v. Let $\xi^* \in \mathcal{R}^n$ be a vector satisfying $\sigma^1(\phi_0)'\left[I - \mathbf{P}^0\right]\xi^* > 0$, which exists since $\sigma^1(\phi_0)'\left[I - \mathbf{P}^0\right] \neq \mathbf{0}'$ by Assumption 2. Let

$$\xi = \frac{2}{\sigma^1(\phi_0)'\left[I - \mathbf{P}^0\right]\xi^*}\xi^*,$$

and let $\epsilon^v = \left\|\sigma^1(\phi^v)'\left[I - \mathbf{P}^v\right] - \sigma^1(\phi_0)'\left[I - \mathbf{P}^0\right]\right\|$. Since $\mathbf{P}^v$ converges to $\mathbf{P}^0$, $\epsilon^v \to 0$. Define

$$q_1^v = \frac{\left[I - \mathbf{P}^v\right]\left[q_1^0 + \epsilon^v\xi\right]}{\left\|\left[I - \mathbf{P}^v\right]\left[q_1^0 + \epsilon^v\xi\right]\right\|},$$

which converges to q_1^0. We show that thus-constructed q_1^v belongs to $D_1(\phi^v)$. First, $F_1(\phi^v)q_1^v = \mathbf{0}$ holds since $F_1(\phi^v)\left[I - \mathbf{P}^v\right] = O$. Second, as for the sign normalization restriction, we have

$$\begin{aligned}
\sigma^1(\phi^v)' q_1^v &= \frac{1}{\left\|\left[I-\mathbf{P}^v\right]\left[q_1^0+\epsilon^v\xi\right]\right\|}\left\{\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]q_1^0+\epsilon^v\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi\right\} \\
&= \frac{1}{\left\|\left[I-\mathbf{P}^v\right]\left[q_1^0+\epsilon^v\xi\right]\right\|}\left\{\begin{array}{c}\left(\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]-\sigma^1(\phi_0)'\left[I-\mathbf{P}^0\right]\right)q_1^0 \\ +\epsilon^v\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi\end{array}\right\} \\
&\geq \frac{1}{\left\|\left[I-\mathbf{P}^v\right]\left[q_1^0+\epsilon^v\xi\right]\right\|}\left\{-\epsilon^v\left\|q_1^0\right\|+\epsilon^v\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi\right\} \\
&= \frac{\epsilon^v}{\left\|\left[I-\mathbf{P}^v\right]\left[q_1^0+\epsilon^v\xi\right]\right\|}\left\{\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi-1\right\},
\end{aligned}$$

where the second line follows since $\left[I - \mathbf{P}^0\right] q_1^0 = \mathbf{0}$. By the construction of ξ, $\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi$ converges to 2, so that $\sigma^1(\phi^v)'\left[I-\mathbf{P}^v\right]\xi - 1 > 0$ holds for all large v. Hence, we have $\sigma^1(\phi^v)' q_1^v \geq 0$ for all large v. ■

Proof of Lemma B.2. *Usc*: Let $Q_{1:(i-1)}^v \in \mathcal{Q}_{1:(i-1)}(\phi^v|F)$ be a sequence. By continuity of $\mathcal{Q}_{1:(i-1)}(\phi^v|F)$, $Q_{1:(i-1)}^v$ has a convergent subsequence with limit $Q_{1:(i-1)}^0 \in \mathcal{Q}_{1:(i-1)}(\phi_0|F)$. Let $q_i^v \in D_i(\phi^v, Q_{1:(i-1)}^v)$ and $Q_{1:i}^v = \left[Q_{1:(i-1)}^v, q_i^v\right]$. Since q_i^v has a converging subsequence with limit q_i^0, $Q_{1:i}^0 = \left[Q_{1:(i-1)}^0, q_i^0\right]$ the limit of $Q_{1:i}^v$ exists. Note that $\begin{pmatrix} F_i(\phi^v) \\ Q_{1:(i-1)}^{v\prime} \end{pmatrix} q_i^v = \mathbf{0}$ and $\sigma^i(\phi^v)' q_i^v \geq 0$ holds for all v by the construction of q_i^v, and they hold at the limit as well. Hence, $q_i^0 \in D_i(\phi_0, Q_{1:(i-1)}^0)$ holds, and $Q_{1:i}^0 \in \mathcal{Q}_{1:i}(\phi_0|F)$.

Lsc: Let $Q_{1:i}^0 = \left[Q_{1:(i-1)}^0, q_i^0\right] \in \mathcal{Q}_{1:i}(\phi_0|F)$, and let $k = rank\begin{pmatrix} F_i(\phi_0) \\ Q_{1:(i-1)}^{0\prime} \end{pmatrix}$. By continuity of $\mathcal{Q}_{1:(i-1)}(\phi|F)$, there exists $Q_{1:(i-1)}^v \in \mathcal{Q}_{1:(i-1)}(\phi^v|F)$ converging to $Q_{1:(i-1)}^0$, and by Assumption 1, we can take $Q_{1:(i-1)}^v$ to satisfy $k = rank\begin{pmatrix} F_i(\phi^v) \\ Q_{1:(i-1)}^{v\prime} \end{pmatrix}$ for all v. Let $\mathbf{P}^0$ be the projection matrix that projects $x \in \mathcal{R}^n$ onto the k dimensional linear subspace formed by the row vectors of $\begin{pmatrix} F_i(\phi_0) \\ Q_{1:(i-1)}^{0\prime} \end{pmatrix}$, and $\mathbf{P}^v$ be the projection matrix that projects $x \in \mathcal{R}^n$ onto the k-dimensional linear subspace formed by the row vectors of $\begin{pmatrix} F_i(\phi^v) \\ Q_{1:(i-1)}^{v\prime} \end{pmatrix}$. Then, since $\begin{pmatrix} F_i(\phi^v) \\ Q_{1:(i-1)}^{v\prime} \end{pmatrix} \to \begin{pmatrix} F_i(\phi_0) \\ Q_{1:(i-1)}^{0\prime} \end{pmatrix}$, we have convergence of projection matrices $\mathbf{P}^v \to \mathbf{P}^0$. Following the proof of Lemma B.1, let $\xi^* \in \mathcal{R}^n$ be a vector satisfying $\sigma^i(\phi_0)'\left[I - \mathbf{P}^0\right]\xi^* > 0$, which exists since $\sigma^i(\phi_0)'\left[I - \mathbf{P}^0\right] \neq \mathbf{0}'$ by Assumption 2. Construct q_i^v in the exactly same manner as q_1^v was constructed in the proof of Lemma B.1.

$$\begin{aligned}
q_i^v &= \frac{\left[I-\mathbf{P}^v\right]\left[q_i^0+\epsilon^v\xi\right]}{\left\|\left[I-\mathbf{P}^v\right]\left[q_i^0+\epsilon^v\xi\right]\right\|}, \\
\xi &= \frac{2}{\sigma^i(\phi_0)'\left[I-\mathbf{P}^0\right]\xi^*}\xi^*, \\
\epsilon^v &= \left\|\sigma^i(\phi^v)'\left[I-\mathbf{P}^v\right]-\sigma^i(\phi_0)'\left[I-\mathbf{P}^0\right]\right\|.
\end{aligned}$$

Repeating the same steps as in the proof of Lemma B.1, we have $q_i^v \in D_i(\phi^v, Q_{1:(i-1)}^v)$ and $q_i^v \to q_i^0$.
■

B.2 Zero and Sign Restrictions

In this section, we show continuity of the identified set correspondence in the presence of both zero and sign restrictions. As in Lemma 4.2, we consider the situation, where the sign restrictions are placed only for the j-th shock, i.e., $\mathcal{I}_S = \{j\}$. In this case, the set of feasible $Q_{1:j} = [q_1, \ldots, q_j]$ can be expressed as

$$\mathcal{Q}_{1:j}(\phi|F,S) \equiv \left\{ \left[Q_{1:(i-1)}, q_j\right] : \begin{pmatrix} F_j(\phi) \\ Q'_{1:(i-1)} \end{pmatrix} q_j = \mathbf{0},\, S_j(\phi) q_j \geq 0,\, Q_{1:(i-1)} \in \mathcal{Q}_{1:(i-1)}(\phi|F) \right\}, \tag{B.1}$$

where $\mathcal{Q}_{1:(i-1)}(\phi|F)$ is as defined in (A.1).

On $\mathcal{R}^n$, $x \geq y$ means $x_i \geq y_i$, $i = 1, \ldots, n$, and $x \gg y$ means $x_i > y_i$, $i = 1, \ldots, n$. The next proposition characterizes a sufficient condition for the impulse response identified set to be a continuous correspondence. Note that the requirement for the number of zero restrictions considered in the next proposition is slightly weaker than that of Lemma 4.2.

Proposition B.2 *Let $r = c_{\tilde{i}h}(\phi) q_j$ be an impulse response of interest. Suppose $\mathcal{I}_S = \{j\}$, i.e., the sign restrictions are placed only for the impulse responses to the j-th structural shock. Suppose that the zero restrictions $F(\phi, Q) = 0$ satisfy $f_i \leq n - i$ for all $i = 1, \ldots, n$. Assume that the following four conditions hold.*

1. *$\mathcal{Q}_{1:(i-1)}(\phi|F)$ is a continuous correspondence at $\phi = \phi_0$.*
2. *$\mathcal{Q}_{1:j}(\phi|F,S)$ defined in (B.1) is non-empty for all $\phi \in G$ in an open neighborhood of ϕ_0.*
3. *Let $\mathcal{K}_j(\phi)$ be as defined in Assumption 1 of Lemma B.2. $\mathcal{K}_j(\phi) = \mathcal{K}_j(\phi_0)$ for all ϕ in an open neighborhood of ϕ_0.*
4. *Each row vector of $\begin{pmatrix} S_j(\phi_0) \\ \sigma^j(\phi_0)' \end{pmatrix}$ is linearly independent of the row vectors of $\begin{pmatrix} F_j(\phi_0) \\ Q'_{1:(j-1)} \end{pmatrix}$ for every $Q_{1:(j-1)} \in \mathcal{Q}_{1:(j-1)}(\phi|F)$.*

Then, the impulse response identified set $IS_r(\phi|F,S)$ is a continuous correspondence at $\phi = \phi_0$ for all $\tilde{i} = 1, \ldots, n$ and $h = 0, 1, 2 \ldots$.

Proof. We first prove continuity of correspondence $\mathcal{Q}_{1:j}(\phi|F,S)$ under the given assumptions. Usc of $\mathcal{Q}_{1:j}(\phi|F,S)$ can be shown by replicating the proof of the usc part in Lemma B.2. Lsc of $\mathcal{Q}_{1:j}(\phi|F,S)$ can be shown by slightly modifying the proof of lsc in Lemma B.2.

Let $Q^0_{1:j} = \left[Q^0_{1:(j-1)}, q^0_j\right] \in \mathcal{Q}_{1:j}(\phi_0|F,S)$, and let $k = rank \begin{pmatrix} F_j(\phi_0) \\ Q^{0\prime}_{1:(j-1)} \end{pmatrix}$. By continuity of $\mathcal{Q}_{1:(j-1)}(\phi|F)$ as assumed, there exists $Q^v_{1:(j-1)} \in \mathcal{Q}_{1:(j-1)}(\phi^v|F)$ converging to $Q^0_{1:(j-1)}$, and by Assumption 3, we can take $Q^v_{1:(j-1)}$ to satisfy $k = rank \begin{pmatrix} F_j(\phi^v) \\ Q^{v\prime}_{1:(j-1)} \end{pmatrix}$ for all v. Let $\mathbf{P}^v$ be the sequence of projection matrices $\mathbf{P}^v \to \mathbf{P}^0$, where $\mathbf{P}^v$ and $\mathbf{P}^0$ are as defined in the proof of Lemma B.2. Let $\xi^* \in \mathcal{R}^n$ be a vector satisfying $\begin{pmatrix} S_j(\phi_0) \\ \sigma^j(\phi_0)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \xi^* \gg \mathbf{0}$, which is assumed to exist by Assumption 4, and let $\eta = \max \left\{ \begin{pmatrix} S_j(\phi_0) \\ \sigma^j(\phi_0)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \xi^* \right\} > 0$. Analogous to the proof of Lemma B.2, consider constructing q^v_j as

$$
\begin{aligned}
q^v_j &= \frac{\left[I - \mathbf{P}^v\right]\left[q^0_j + \epsilon^v \xi\right]}{\left\| \left[I - \mathbf{P}^v\right]\left[q^0_j + \epsilon^v \xi\right] \right\|}, \\
\xi &= \frac{2}{\eta}\xi^*, \\
\epsilon^v &= \left\| \begin{pmatrix} S_j(\phi^v) \\ \sigma^j(\phi^v)' \end{pmatrix} \left[I - \mathbf{P}^v\right] - \begin{pmatrix} S_j(\phi_0) \\ \sigma^j(\phi_0)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \right\|.
\end{aligned}
$$

By construction $q^v_j \to q^0_j$, and $\begin{pmatrix} F_j(\phi^v) \\ Q'_{1:(j-1)} \end{pmatrix} q^v_j = 0$ hold. Note also that $\begin{pmatrix} S_j(\phi^v) \\ \sigma^j(\phi^v)' \end{pmatrix} q^v_j \geq \mathbf{0}$ holds for all large v, since, for $a(\phi)'$, an arbitrary row vector of $\begin{pmatrix} S_j(\phi) \\ \sigma^j(\phi)' \end{pmatrix}$, we have

$$
\begin{aligned}
a(\phi^v)' q^v_j &= \frac{1}{\left\| \left[I - \mathbf{P}^v\right]\left[q^0_1 + \epsilon^v \xi\right] \right\|} \left\{ a(\phi^v)' \left[I - \mathbf{P}^v\right] q^0_1 + \epsilon^v a(\phi^v)' \left[I - \mathbf{P}^v\right] \xi \right\} \\
&= \frac{1}{\left\| \left[I - \mathbf{P}^v\right]\left[q^0_1 + \epsilon^v \xi\right] \right\|} \left\{ \begin{array}{c} \left(a(\phi^v)' \left[I - \mathbf{P}^v\right] - a(\phi_0)' \left[I - \mathbf{P}^0\right] \right) q^0_j \\ + \epsilon^v a(\phi^v)' \left[I - \mathbf{P}^v\right] \xi \end{array} \right\} \\
&\geq \frac{1}{\left\| \left[I - \mathbf{P}^v\right]\left[q^0_1 + \epsilon^v \xi\right] \right\|} \left\{ -\epsilon^v \left\| q^0_j \right\| + \epsilon^v a(\phi^v)' \left[I - \mathbf{P}^v\right] \xi \right\} \\
&= \frac{\epsilon^v}{\left\| \left[I - \mathbf{P}^v\right]\left[q^0_1 + \epsilon^v \xi\right] \right\|} \left\{ a(\phi^v)' \left[I - \mathbf{P}^v\right] \xi - 1 \right\},
\end{aligned}
$$

where the last expression is nonnegative for all large v, provided that $a(\phi^v)' \left[I - \mathbf{P}^v\right] \xi$ converges to $a(\phi_0)' \left[I - \mathbf{P}^0\right] \xi > 0$. Hence, $Q^v_{1:j} = \left[Q^v_{1:(j-1)}, q^v_j\right] \in \mathcal{Q}_{1:j}(\phi^v|F,S)$ holds and we conclude $\mathcal{Q}_{1:j}(\phi|F,S)$ is lsc at $\phi = \phi_0$.

The impulse response identified set can be seen as a product correspondence in the form of

$$
IS_r(\phi|F,S) \equiv r(\phi, \mathcal{Q}_{1:j}(\phi|F,S)) \equiv \left\{ c_{\tilde{i}h}(\phi)' q_j : Q_{1:j} \in \mathcal{Q}_{1:j}(\phi|F,S) \right\}.
$$

Following the same argument as in the proof of Lemma 3 (iv) of Moon et al (2013), we conclude $IS_r(\phi|F,S)$ is a continuous correspondence at $\phi = \phi_0$. ■

References

Aruoba, B., and F. Schorfheide (2011): "Sticky Prices versus Monetary Frictions: An Estimation of Policy Trade-offs," *American Economic Journal: Macroeconomics*, 3, 60–90.

Aubin, J.-P., and H. Frankowska (1990): *Set-Valued Analysis*. BirkhÃďuser.

Baumeister, C., and J. Hamilton (2013): "Sign Restrictions, Structural Vector Autoregressions, and Useful Prior Information," *unpublished manuscript*.

Berger, J., and L. Berliner (1986): "Robust Bayes and Empirical Bayes Analysis with Îţ-contaminated Priors," *The Annals of Statistics*, 14, 461–486.

Blanchard, O., and D. Quah (1993): "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review*, 83, 655–673.

Border, K. (2013): "Introduction to Correspondences," *unpublished manuscript*.

Canova, F., and G. D. Nicolo (2002): "Monetary Disturbances Matter for Business Fluctuations in the G-7," *Journal of Monetary Economics*, 49, 1121–1159.

Chamberlain, G., and E. Leamer (1976): "Matrix Weighted Averages and Posterior Bounds,," *Journal of Royal Statistical Society, Series B*, 38, 73–84.

Christiano, L., M. Eichenbaum, and C. Evans (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113, 1–45.

Denneberg, D. (1994): *Non-additive Measures and Integral*. Kluwer Academic Publishers.

DeRobertis, L., and J. Hartigan (1981): "Bayesian Inference Using Intervals of Measures," *The Annals of Statistics*, 9, 235–244.

Faust, J. (1998): "The Robustness of Identified VAR Conclusions about Money," *Carnegie-Rochester Conference Series on Public Policy*, 48, 207–244.

Fry, R., and A. Pagan (2011): "Sign Restrictions in Structural Vector Autoregressions: A Critical Review," *Journal of Economic Literature*, 49, 938–960.

Gordon, S., and D. Boccanfuso (2001): "Learning from Structural Vector Autoregression Models," *unpublished manuscript*.

Kadane, J. (1974): "The Role of Identification in Bayesian Theory," in *Studies in Bayesian Econometrics and Statistics, In Honor of Leonard J. Savage*, ed. by A. Fienberg, S.E. & Zellner. North Holland, Amsterdam.

Kitagawa, T. (2012): "Estimation and Inference for Set-identified Parameters Using Posterior Lower Probabilities," *unpublished manuscript*.

KLEIN, B., AND E. TAMER (2013): "Default Bayesian Inference in a Class of Partially Identified Models," *unpublished manuscript*.

LEAMER, E. (1982): "Sets of Posterior Means with Bounded Variance Priors," *Econometrica*, 50, 725–736.

LEEPER, E., C. SIMS, AND T. ZHA (1996): "What does Monetary Policy Do?," *Brookings Papers on Economic Activity*, 2, 1–78.

LIAO, Y., AND A. SIMONI (2013): "Semi-parametric Bayesian Partially Identified Models based on Support Function," *unpublished manuscript*.

MANSKI, C. (2003): *Partial Identification of Probability Distributions*. Springer-Verlag, New York.

MOON, H., AND F. SCHORFHEIDE (2011): "Bayesian and Frequentist Inference in Partially Identified Models," *unpublished manuscript*.

MOON, H., AND F. SHORFHEIDE (2012): "Bayesian and Frequentist Inference in Partially Identified Models," *Econometrica*, 80, 755–782.

MOON, H., F. SHORFHEIDE, AND E. GRANZIERA (2013): "Inference for VARs Identified with Sign Restrictions," *unpublished manuscript*.

POIRIER, D. (1998): "Revising Beliefs in Nonidentified Models," *Econometric Theory*, 14, 483–509.

RUBIO-RAMIREZ, J., D. WAGGONER, AND T. ZHA (2010): "Structural Vector Autoregressions: Theory of Identification and Algorithm for Inference," *Review of Economic Studies*, 77, 665–696.

SIMS, C. (1980): "Macroeconomics and Reality," *Econometrica*, 48, 1–48.

SIMS, C., AND T. ZHA (2006): "Does Monetary Policy Generate Recessions?," *Macroeconomic Dynamics*, 10, 231–272.

SIMS, C.A. & ZHA, T. (1998): "Error Bands for Impulse Responses," *Econometrica*, 67, 1113–1155.

SUNDARAM, R. K. (1996): *A First Course in Optimization Theory*. Cambridge University Press.

UHLIG, H. (2005): "What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics*, 52, 381–419.

WASSERMAN, L. (1990): "Prior Envelopes Based on Belief Functions," *The Annals of Statistics*, 18, 454–464.