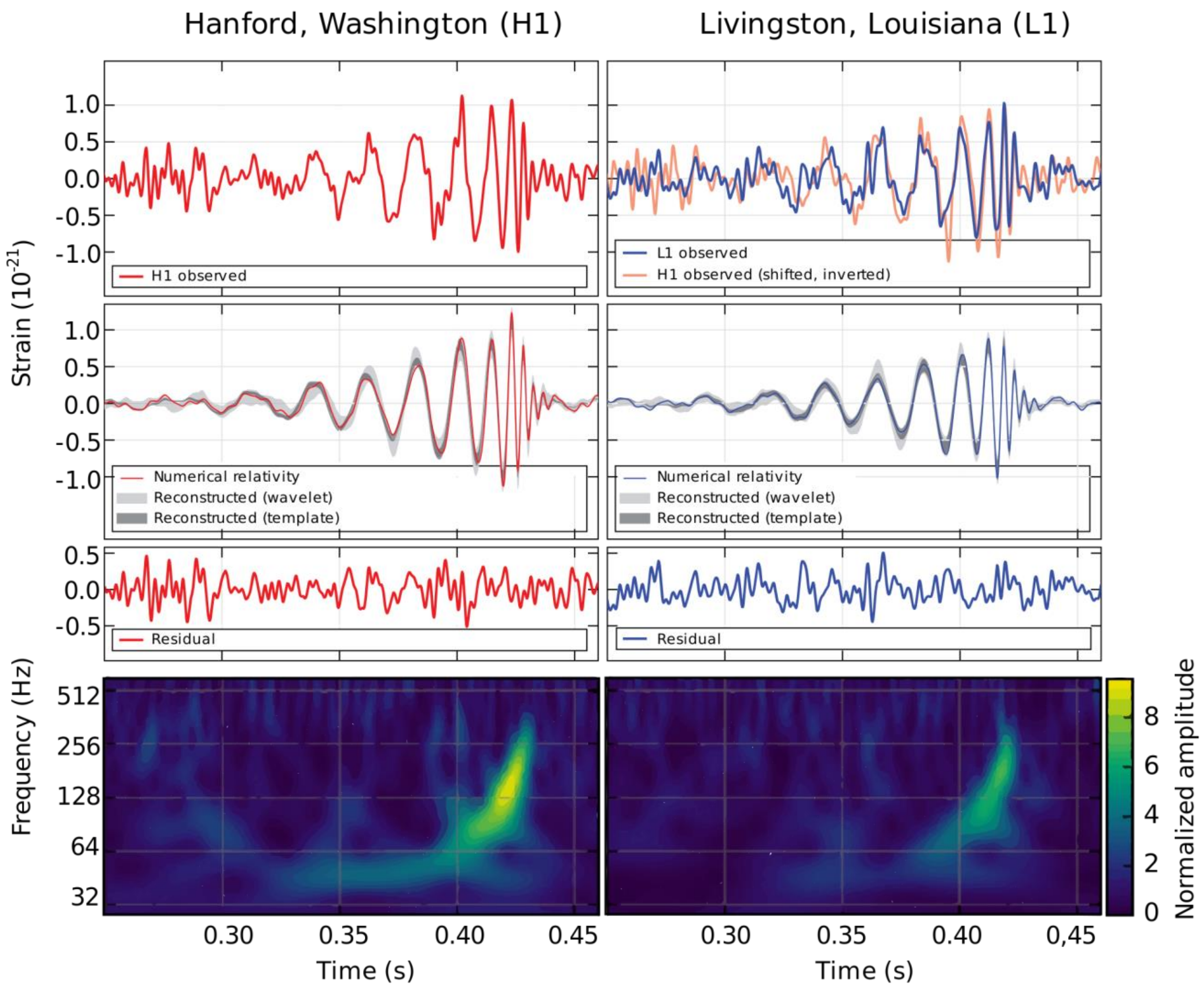
Primordial black holes,

scale invariance, and inflation

Subodh Patil

Niels Bohr International Academy

IAS HKUST, Jan 9 2020



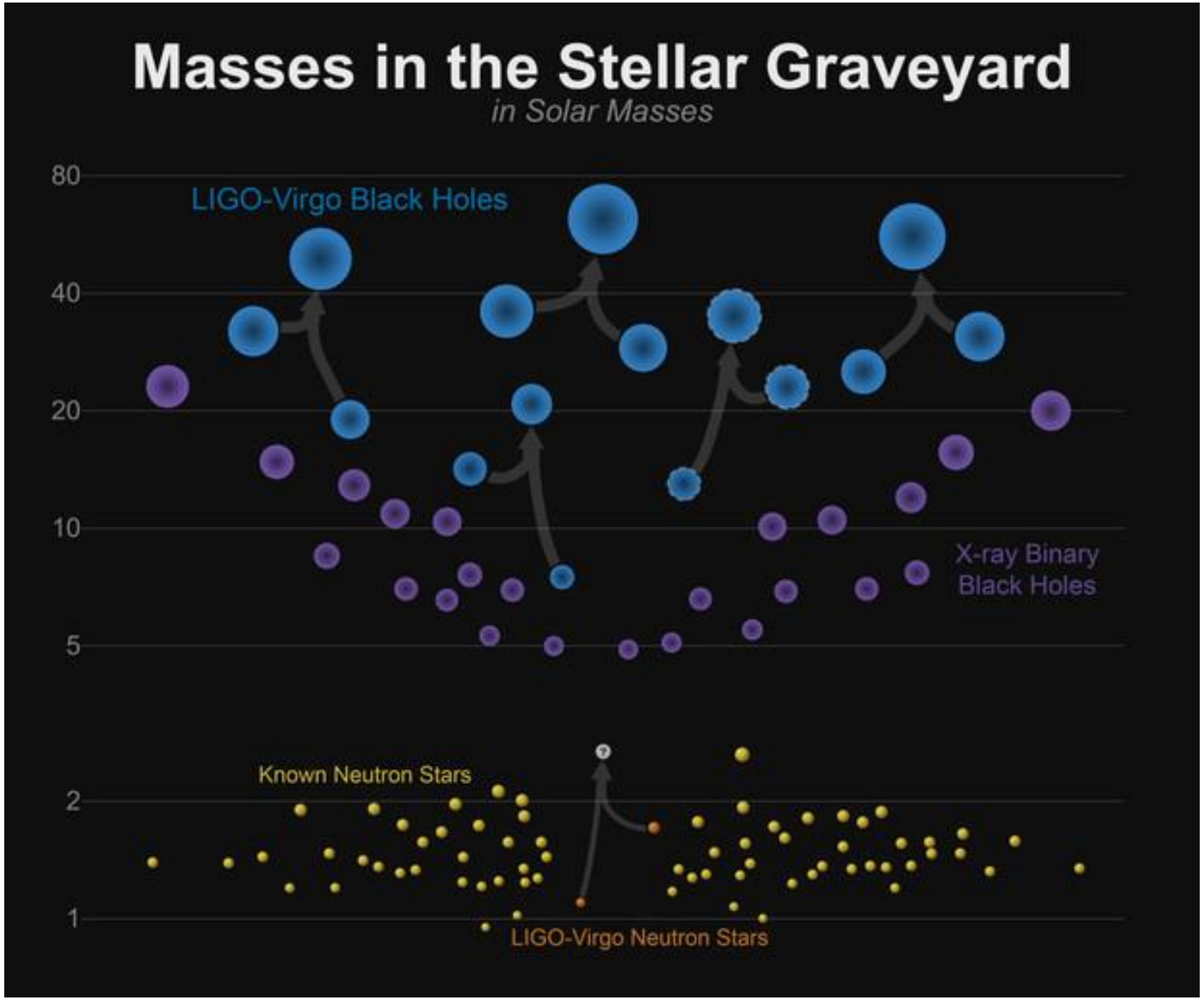
LIGO Collaboration (2016) $\{m_1, m_2\} \sim (29, 36) M_\odot$

Did LIGO detect dark matter?

Simeon Bird,* Ilias Cholis, Julian B. Muñoz, Yacine Ali-Haïmoud, Marc Kamionkowski, Ely D. Kovetz, Alvise Raccanelli, and Adam G. Riess[1]

[1] *Department of Physics and Astronomy, Johns Hopkins University, 3400 N. Charles St., Baltimore, MD 21218, USA*

We consider the possibility that the black-hole (BH) binary detected by LIGO may be a signature of dark matter. Interestingly enough, there remains a window for masses $20\,M_\odot \lesssim M_{\rm bh} \lesssim 100\,M_\odot$ where primordial black holes (PBHs) may constitute the dark matter. If two BHs in a galactic halo pass sufficiently close, they radiate enough energy in gravitational waves to become gravitationally bound. The bound BHs will rapidly spiral inward due to emission of gravitational radiation and ultimately merge. Uncertainties in the rate for such events arise from our imprecise knowledge of the phase-space structure of galactic halos on the smallest scales. Still, reasonable estimates span a range that overlaps the $2-53$ Gpc^{-3} yr^{-1} rate estimated from GW150914, thus raising the possibility that LIGO has detected PBH dark matter. PBH mergers are likely to be distributed spatially more like dark matter than luminous matter and have no optical nor neutrino counterparts. They may be distinguished from mergers of BHs from more traditional astrophysical sources through the observed mass spectrum, their high ellipticities, or their stochastic gravitational wave background. Next generation experiments will be invaluable in performing these tests.


Masses in the Stellar Graveyard
in Solar Masses
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
X-ray Binary Black Holes
Known Neutron Stars
LIGO-Virgo Neutron Stars

Could these black holes be primordial in origin?

- Consistent* with no spin (hard to produce astrophysically)

- Black holes with masses $> 10^{20} M_{\rm pl} \sim 10^{15}$ g $\sim 10^{-18} M_\odot$ can survive the age of the universe without Hawking evaporating.

- Can be produced with masses below the Chandrasekhar limit $\sim 1.4 M_\odot \sim 3 \times 10^{33}$ g (observation of which would be smoking gun proof of their existence!)

Could these black holes be primordial in origin?

Can be produced by:

- Bubble collisions in 1’st order phase transitions;
- cosmic string loop collapse; domain walls;
- collapse of large (primordial?) overdensities;
- In the latter case, abundances can therefore be probed by complimentary observations of small scale power.

Primordial black hole formation

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\bar{\rho}$$

on this background, consider a spherically symmetric perturbation that will eventually collapse into a black hole

Primordial black hole formation

$$ds^2 = -dt^2 + a^2(t)e^{2\psi}\delta_{ij}dx^i dx^j$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dR^2}{1-K(R)R^2} + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

$$K = -\frac{\psi'(r)}{r}\frac{2+r\psi'(r)}{e^{2\psi(r)}}$$

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$$R^{(3)} = -\frac{e^{-2\psi}}{3a^2}\delta^{ij}\left[2\partial_i\partial_j\psi + \partial_i\psi\partial_j\psi\right] = \frac{K}{a^2}\left(1 + \frac{d\ln K(R)}{3d\ln R}\right).$$

$$H^2 + \frac{K(r)}{a^2} = \frac{8\pi G}{3}\rho \qquad\qquad \Delta := \frac{\rho-\bar{\rho}}{\bar{\rho}} = \frac{3K}{8\pi G\bar{\rho}a^2} = \frac{K}{H^2a^2}.$$

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$$\Delta \sim 1 \leftrightarrow H = 0$$

Primordial black hole formation

Harada, Yu, Kohri (2013): a more detailed analysis for uniform density profiles implied $\Delta_c = \sin^2\left(\pi\sqrt{w}/(1+3w)\right)$

(not so accurate as you approach matter domination, more on this later…)

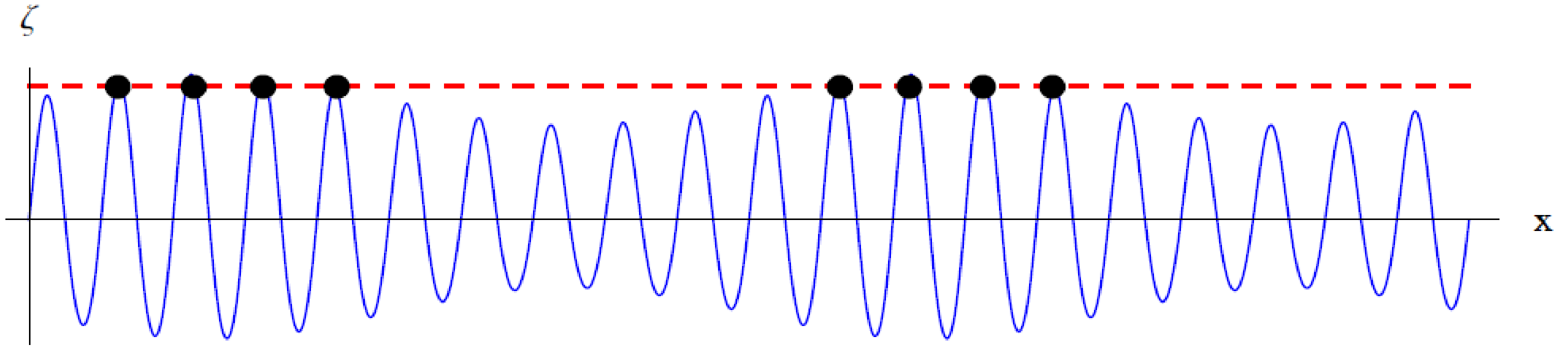
In reality, different density profiles, different thresholds… during *radiation domination* $\Delta_c \sim 0.3 \to 0.66$

(See Kalaja et al, 1908.03596)

Primordial black hole formation

Press-Schecter formalism – $\beta = \int_{\Delta_c}^{\infty} P(\Delta) d\Delta$

β is the mass fraction of PBH's at the time of formation



$$P(\Delta) = \frac{1}{\sqrt{2\pi\langle\Delta\rangle^2}} \exp\left(-\frac{\Delta^2}{2\langle\Delta\rangle^2}\right) \qquad \langle\Delta^2\rangle = \int_0^{\infty} \frac{dk}{k} \mathcal{P}_\Delta(k)$$

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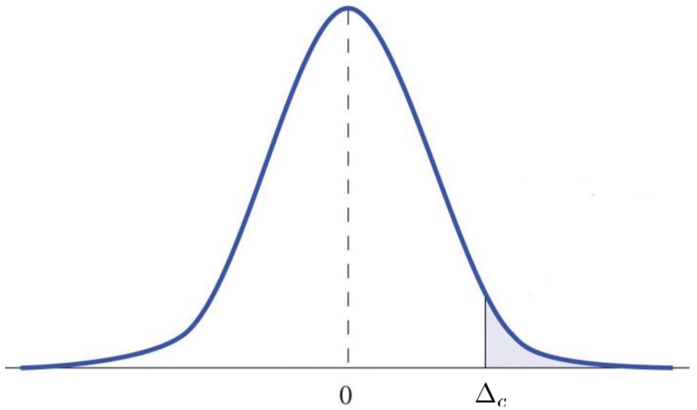

$$\mathcal{P}_\Delta(k) = \frac{4(1+w)^2}{(5+3w)^2} \left(\frac{k}{aH}\right)^4 \mathcal{P}_\mathcal{R}(k)$$

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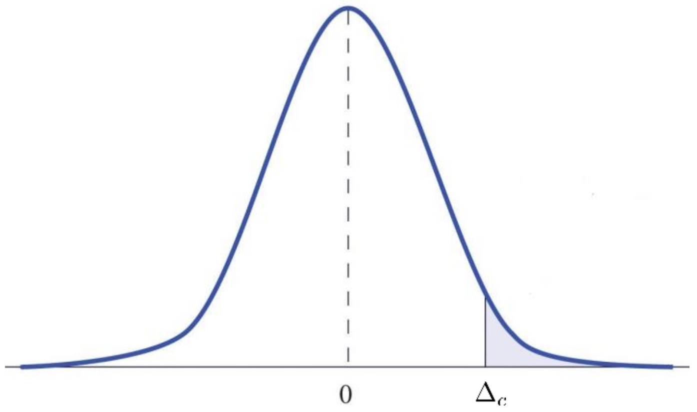

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power spectrum of curvature perturbation from inflation

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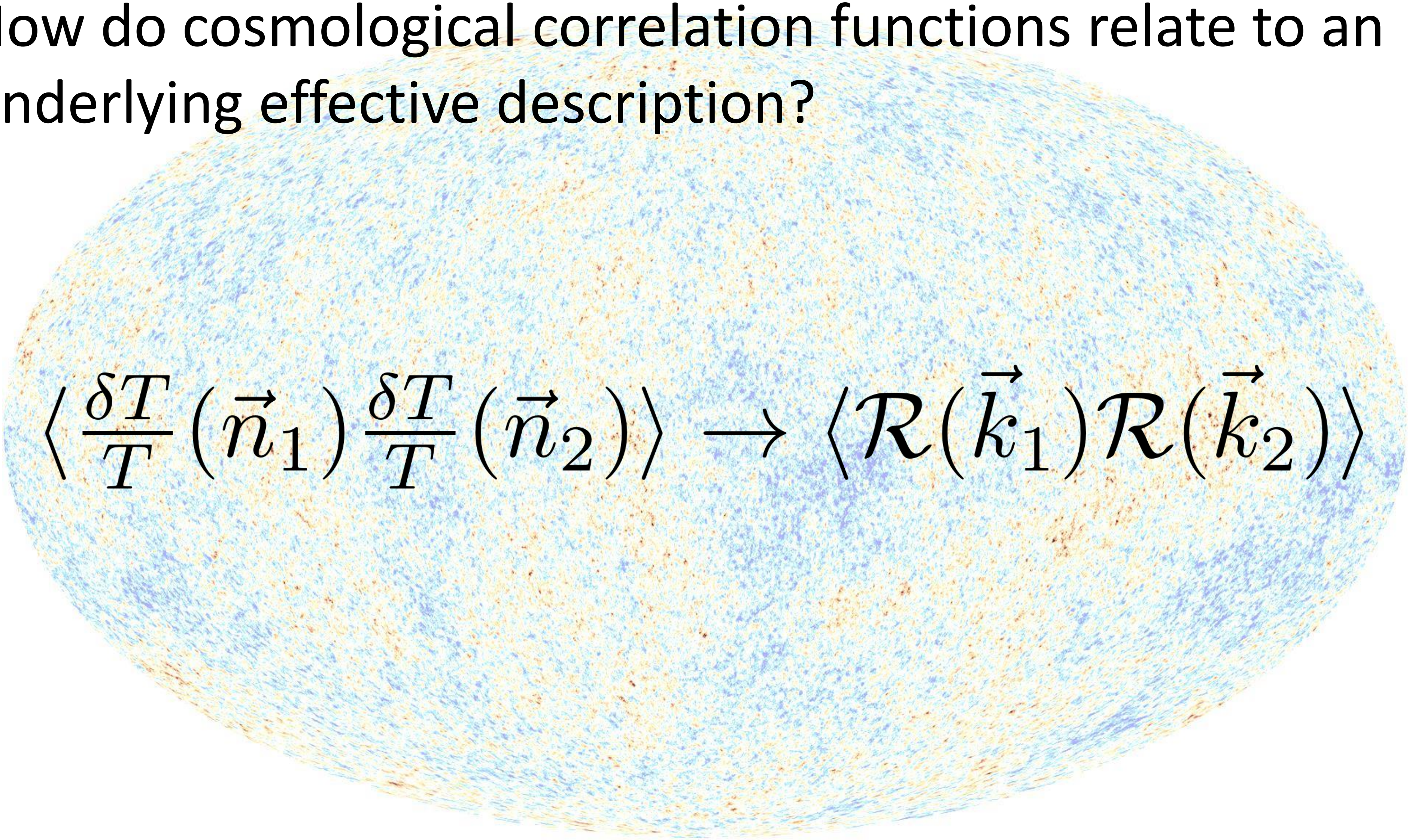
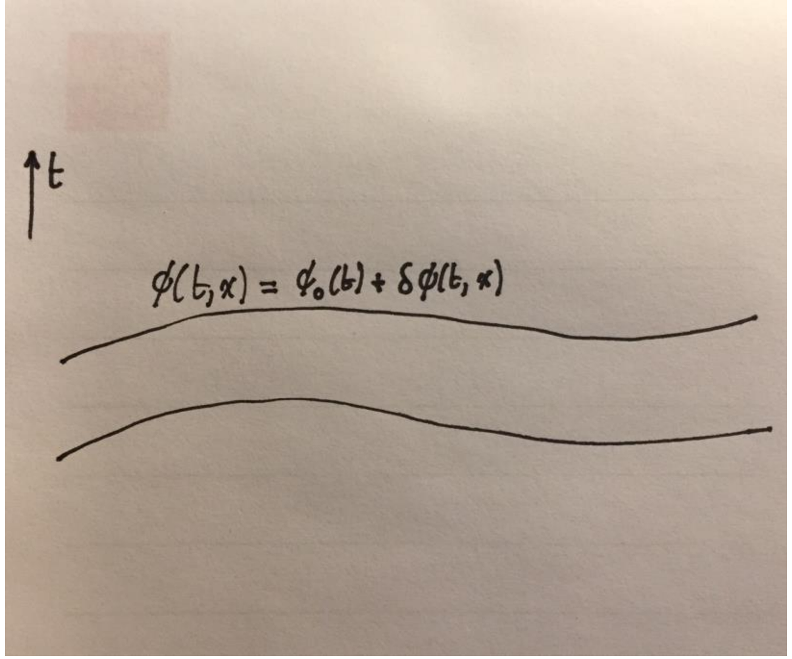
How do cosmological correlation functions relate to an underlying effective description?



$$\langle \frac{\delta T}{T}(\vec{n}_1)\frac{\delta T}{T}(\vec{n}_2)\rangle \rightarrow \langle \mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\rangle$$

How do cosmological correlation functions relate to an underlying effective description? (Cheung et al. 2007)

$$\mathcal{L}_{\rm tot} = \frac{M_{\rm pl}^2}{2}R + ... + \mathcal{L}(\phi, \nabla\phi, \nabla^2\phi, ...)$$



$$\phi(x,t) = \phi_0(t) + \delta\phi(t,x)$$

$$t \;\rightarrow\; t + \pi$$

$$\phi_0 + \delta\phi(t,x) \rightarrow \phi_0 + \delta\phi(t,x) - \dot{\phi}_0\pi \equiv \phi_0(t)$$

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$$

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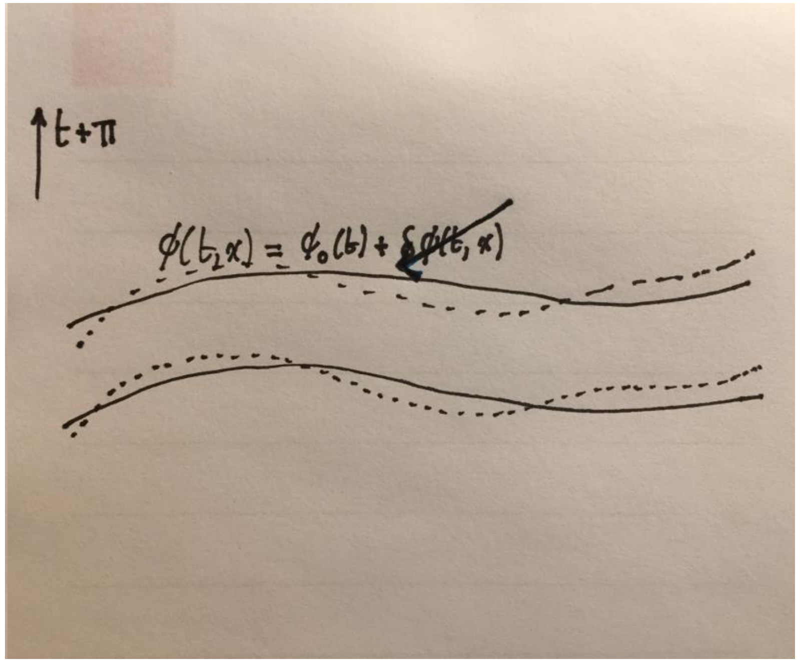

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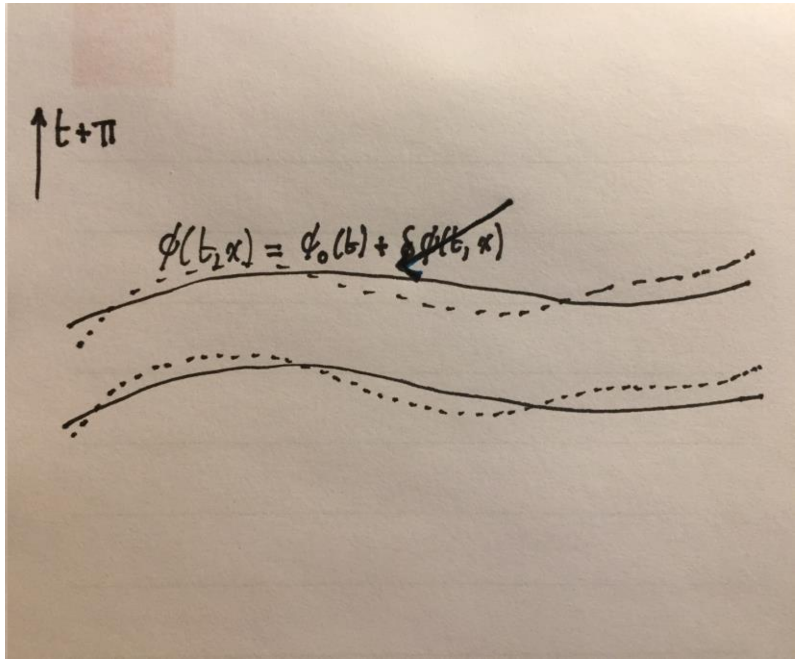
How do cosmological correlation functions relate to an underlying effective description? (Cheung et al. 2007)

Q) Where did the scalar perturbation go?



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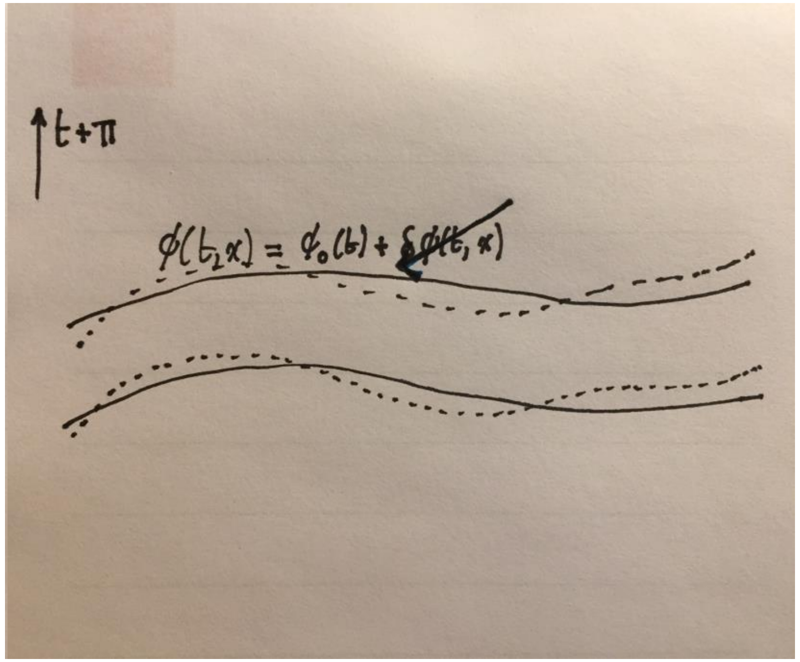
How do cosmological correlation functions relate to an underlying effective description? (Cheung et al. 2007)

A) It got `eaten' by the metric, which now propagates a longitudinal polarization…



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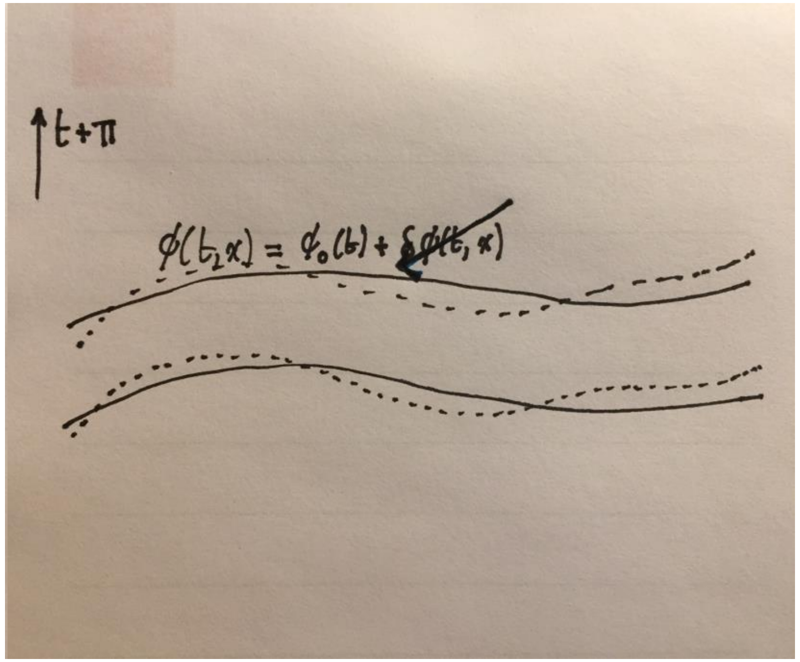
Since $\mathcal{R}$ is a Goldstone, $\mathcal{R}$ = const. will *always* be a solution for $k \ll 1$ to any order in perturbation theory, since only derivative interactions. This is what imprints anisotropies on the CMB... (Bond, Salopek 1990; Assasi, Baumann, Green 2012)



$$S_2 = \int d^4x a^3 \epsilon M_{\rm pl}^2 \left(\frac{\dot{\mathcal{R}}^2}{c_s^2} - \frac{(\partial\mathcal{R})^2}{a^2} + \mu^{-2}\frac{(\partial^2\mathcal{R})^2}{a^4} \right)$$

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \frac{1}{c_s^2} = 1 - \frac{2M_2^4}{M_{\rm pl}^2 \dot{H}}$$

changes to zero and two derivative terms in the parent theory manifest here...

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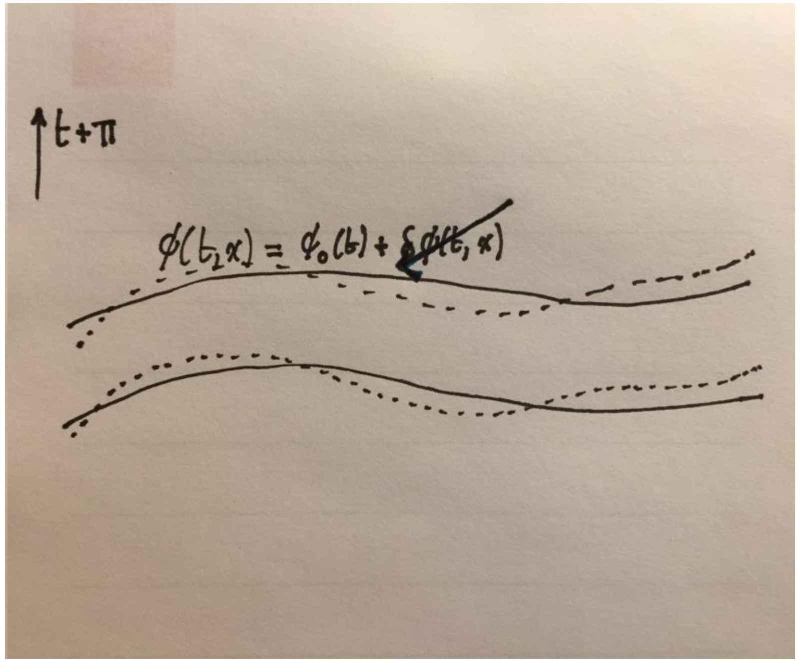

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changes to two and four derivative terms in the parent theory manifest here…

$$2\pi^2\delta^3(\vec{k}_1+\vec{k}_2)\mathcal{P}_{\mathcal{R}}(k_1) = k_1^3\langle\mathcal{R}(\vec{k}_1)\mathcal{R}(\vec{k}_2)\rangle$$

$$|\Psi\rangle = |0\rangle$$

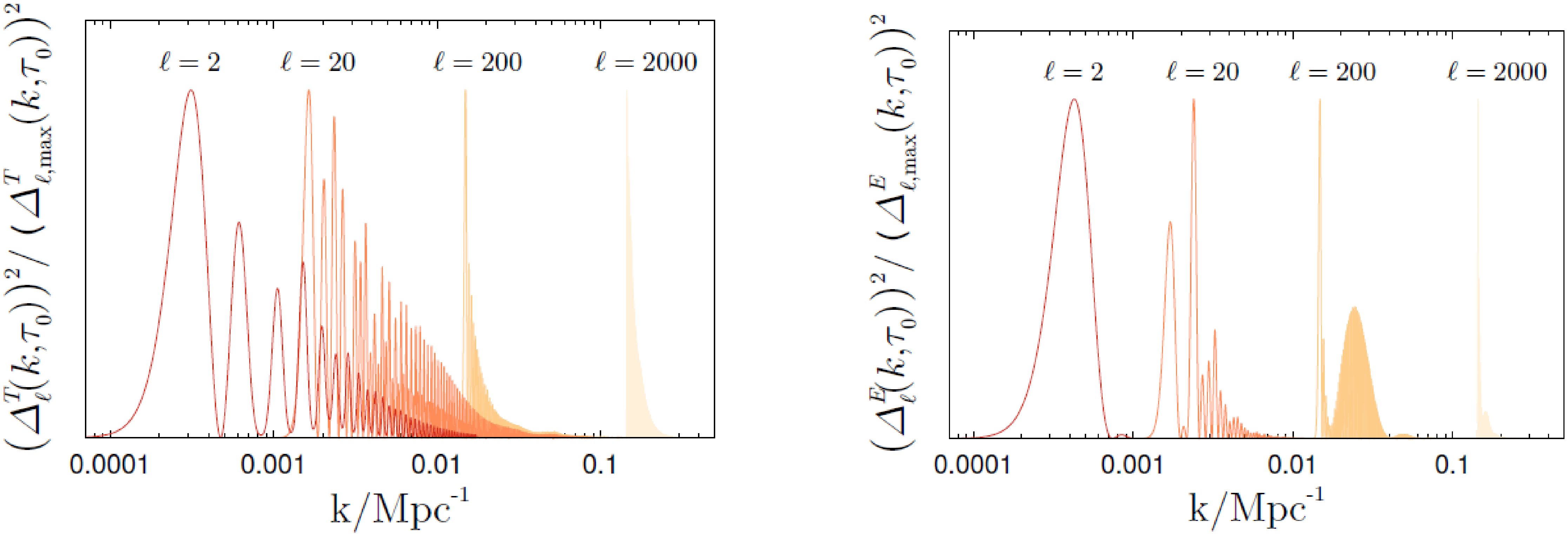
$$\frac{\Delta\widehat{T}}{\widehat{T}}(\vec{n}) = \sum_{\ell m} a^T_{\ell m} Y_{\ell m}(\vec{n})$$

$$\langle a^X_{\ell m} a^{Y*}_{\ell' m'}\rangle = C^{XY}_\ell \delta_{\ell\ell'}\delta_{mm'}$$

$$C^{XY}_\ell = \frac{1}{2\pi^2}\int \mathrm{d}\ln k\,\Delta^X_\ell(k,\tau_0)\Delta^Y_\ell(k,\tau_0)\mathcal{P}_{\mathcal{R}}(k)$$

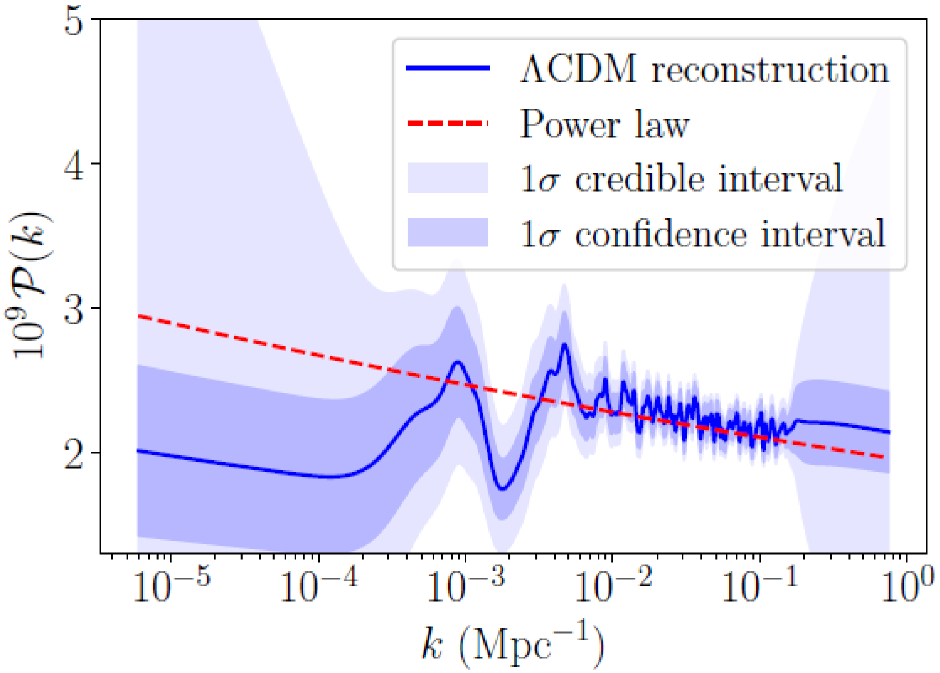
$$\Delta^X_\ell(k,\tau_0) = \int_{\tau_{\mathrm{ini}}}^{\tau_0} \mathrm{d}\tau\; S^X(k,\tau) j_\ell(k(\tau-\tau_0))$$

\`Transfer function’ = non-primordial cosmology + geometry

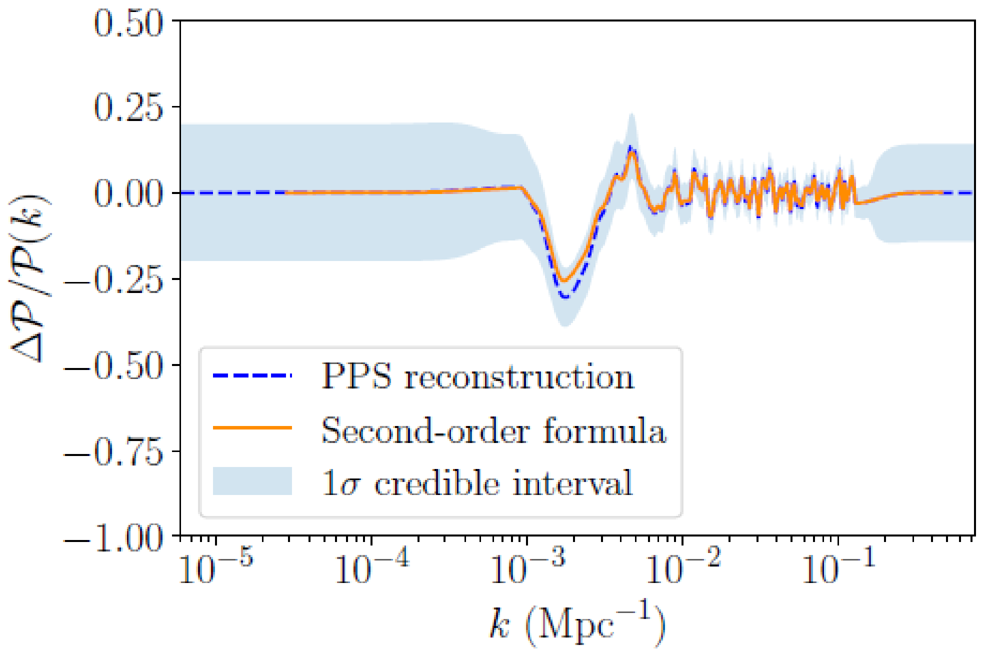


Transfer functions `sample' underlying 2-pt function

(Aside – from reconstructed data to `Wilson functions’)



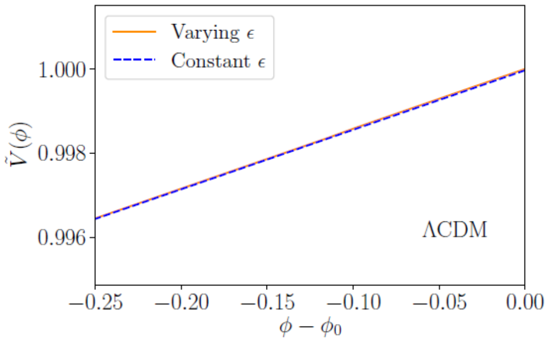
Cosmological parameters	
h	0.667
ω_{b}	0.022
ω_{c}	0.12
z_{re}	9.94

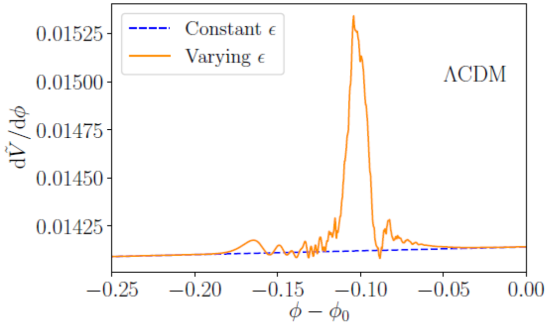


Can `invert’ for EFT parameters given a scale dependent reconstruction to accuracy of order $\left(\frac{\Delta\mathcal{P}}{\mathcal{P}}\right)^3$**…**

$$\frac{\Delta\epsilon}{\epsilon}(\tau) = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}k}{k}\frac{\Delta_1\mathcal{P}_\mathcal{R}}{\mathcal{P}_\mathcal{R}}(k)\left(\frac{2\sin^2(k\tau)}{k\tau} - \sin(2k\tau)\right)$$

$$\frac{\Delta_1\mathcal{P}_\mathcal{R}}{\mathcal{P}_\mathcal{R}}(k) = \frac{1}{2}\left(-1+\sqrt{1+4\frac{\Delta\mathcal{P}_{\mathrm{rec}}}{\mathcal{P}_\mathcal{R}}(k)}\right)$$



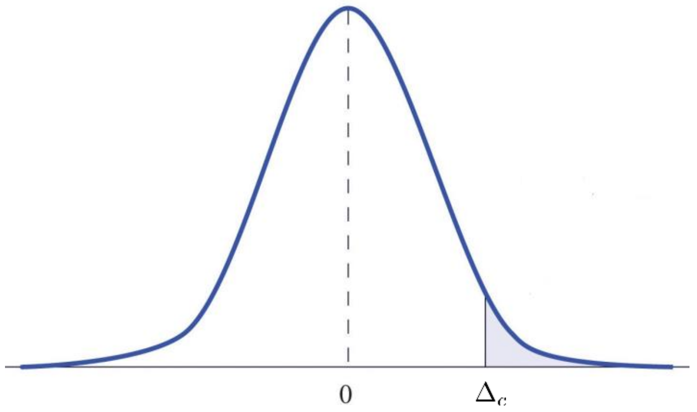


Durakovic, Hunt, Patil, Sarkar (2019)

Primordial black hole formation

Press-Schecter formalism – $\beta = \int_{\Delta_c}^{\infty} P(\Delta) d\Delta \equiv \frac{1}{\gamma} \frac{\rho_{\rm PBH}(M_{\rm f})}{\rho_{\rm tot}}$

β is the mass fraction of PBH's at the time of formation, γ is the fraction of horizon mass to collapse into a black hole…



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power spectrum of curvature perturbation from inflation

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$$\langle\Delta^2\rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\Delta(k)$$

Primordial black hole formation

Critical collapse – PBH mass function given by: (Byrnes et al, 2018)

$$f(M) \equiv \frac{1}{\Omega_{\rm CDM}} \frac{d\Omega_{\rm PBH}}{d\ln M}$$

$$= \frac{1}{\Omega_{\rm CDM}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi\sigma^2(M_{\rm H})}} \exp\left[-\frac{\left(\mu^{1/\gamma}+\Delta_c(M_{\rm H})\right)^2}{2\sigma^2(M_{\rm H})}\right] \frac{M}{\gamma M_{\rm H}} \mu^{1/\gamma} \sqrt{\frac{M_{\rm eq}}{M_{\rm H}}} d\ln M_{\rm H}$$

$$M = \kappa M_{\rm H} \left(\Delta - \Delta_c\right)^{\gamma} \;\; (\kappa = 3.3, \gamma = 0.36, \Delta_c = 0.45) \;\; \mu \equiv \frac{M}{\kappa M_{\rm H}}$$

$$\sigma_R^2 = \int_0^{\infty} \frac{{\rm d}q}{q} \frac{16}{81} \left(q/R\right)^4 \mathcal{P}_{\mathcal{R}}(q) W_R(q)^2; \quad W_R(q)^2 = e^{-(qR)^2}$$

Constants obtained numerically – exact values depend on radial profile presumed…

Primordial black hole formation

During radiation domination (monochromatic):

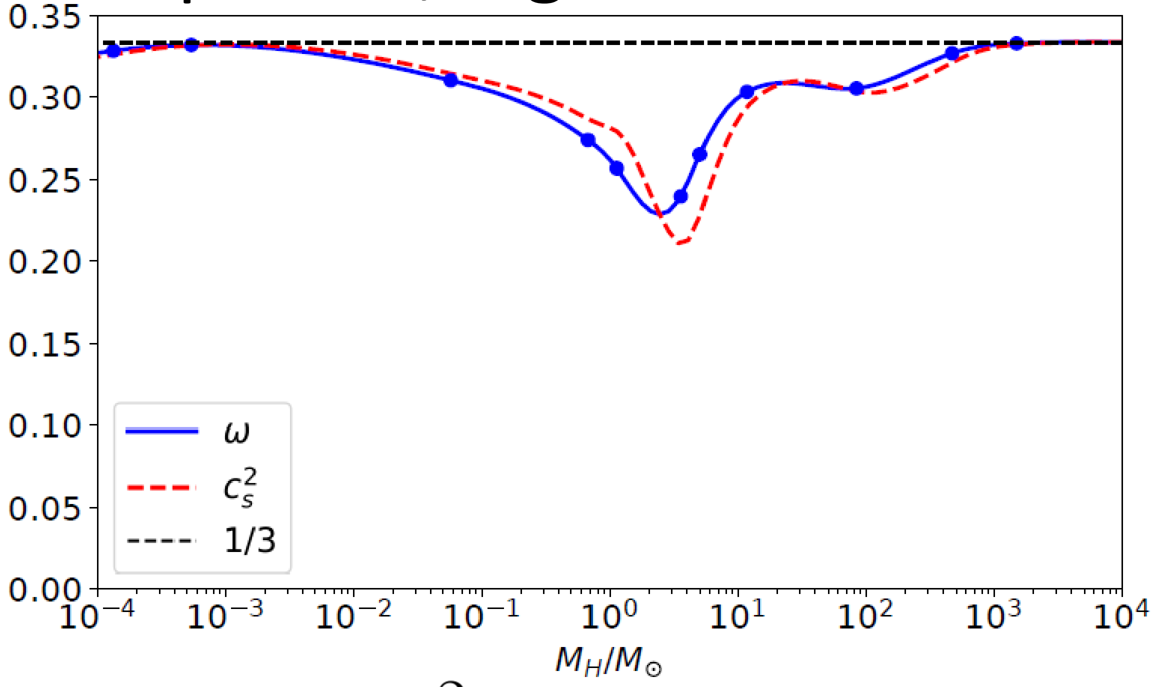
$$f_{\mathrm{PBH}}\left(M_{\mathrm{f}}\right) \simeq \gamma^{3/2}\beta_{\mathrm{RD}}\left(M_{\mathrm{f}}\right)\frac{\Omega_{\mathrm{rad},0}^{3/4}}{\Omega_{\mathrm{DM},0}}\left(\frac{g_{*0}}{g_{*\mathrm{f}}}\right)^{1/4}\left(\frac{M_{\mathrm{H},0}}{M_{\mathrm{f}}}\right)^{1/2}$$

For PBH's to be *all* of DM, we require peak $\mathcal{P}_k \simeq \mathcal{R}^2 \approx 1.26 \cdot 10^{-2}$

N.B. at CMB scales, we have $\mathcal{P}_k \approx 2 \cdot 10^{-9}$ … we must boost power by some seven orders of magnitude (!)

Primordial black hole formation

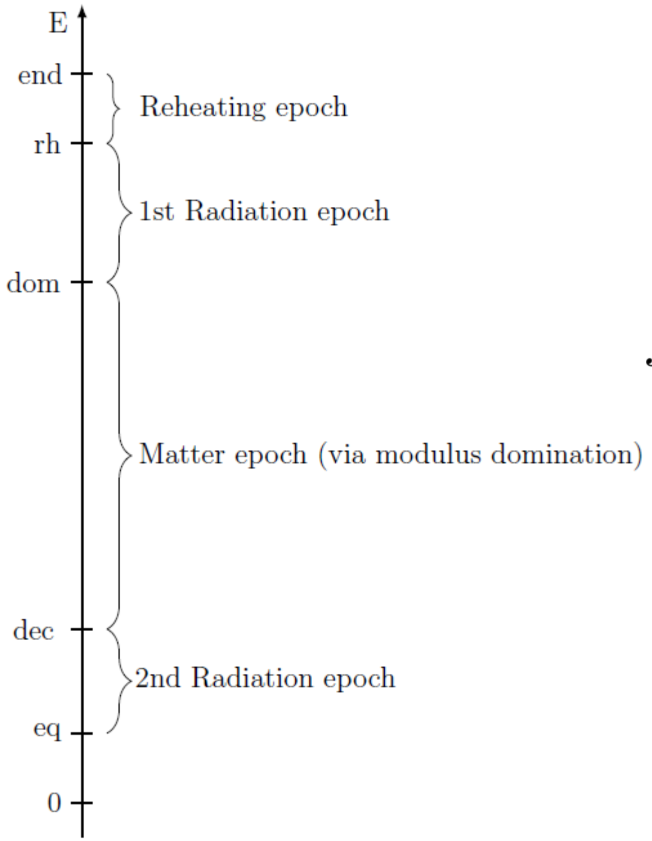
N.B. even the standard thermal history during radiation domination has dips in *w*, e.g. from the QCD crossover:



$$\Delta_c = \sin^2\left(\pi\sqrt{w}/(1+3w)\right)$$

(Fig. from Byrnes, Hindmarsh, Young, Hawkins 2018)

Primordial black hole formation



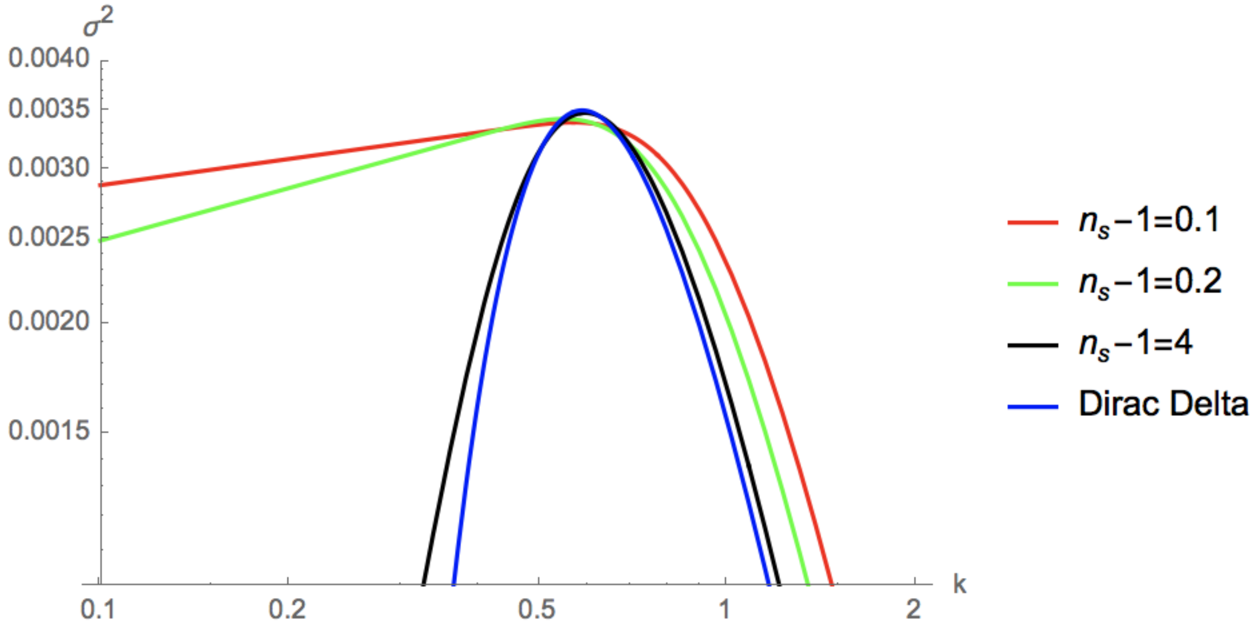
During early matter domination (Cicole, Diaz, Pedro 2018)

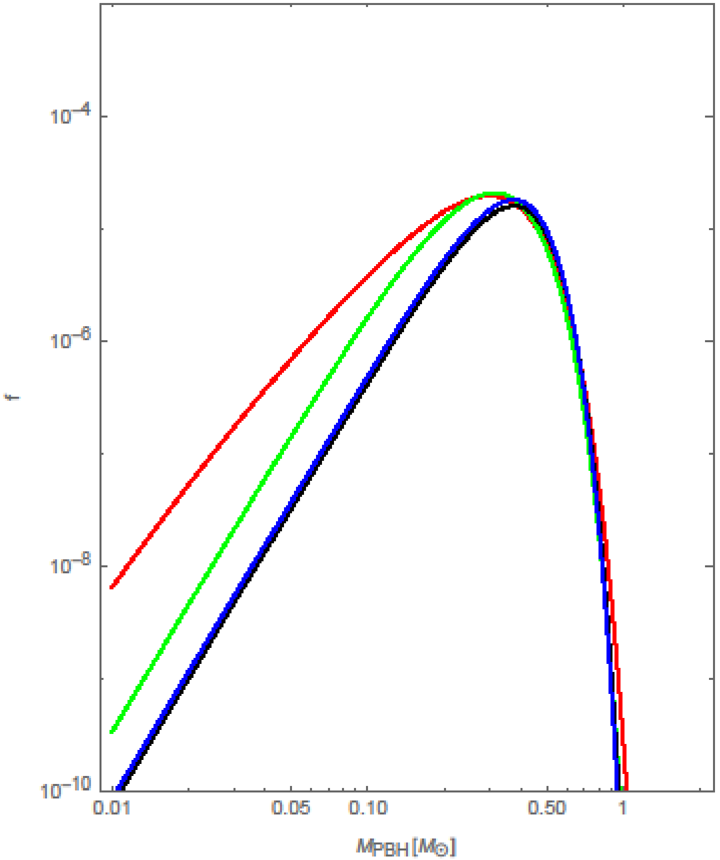
$$f_{\mathrm{PBH}}\left(M_{\mathrm{f}}\right) \simeq \gamma^{3/2} \beta_{\mathrm{MD}}\left(M_{\mathrm{f}}\right) \frac{\Omega_{\mathrm{rad},0}^{3/4}}{\Omega_{\mathrm{DM},0}} \left(\frac{g_{*0}}{g_{*\mathrm{dec}}}\right)^{1/4} \left(\frac{M_{\mathrm{H},0}}{M_{\mathrm{f}}}\right)^{1/2} e^{-\frac{3}{4}\left(N_{\mathrm{MD}}-N_{\mathrm{f}}\right)}$$

For PBH's to be *all* of DM, for $\approx$ GeV mass moduli require only $\mathcal{P}_k \approx 10^{-5}$ (also monochromatic)

a much milder requirement, but requires non-standard thermal history…

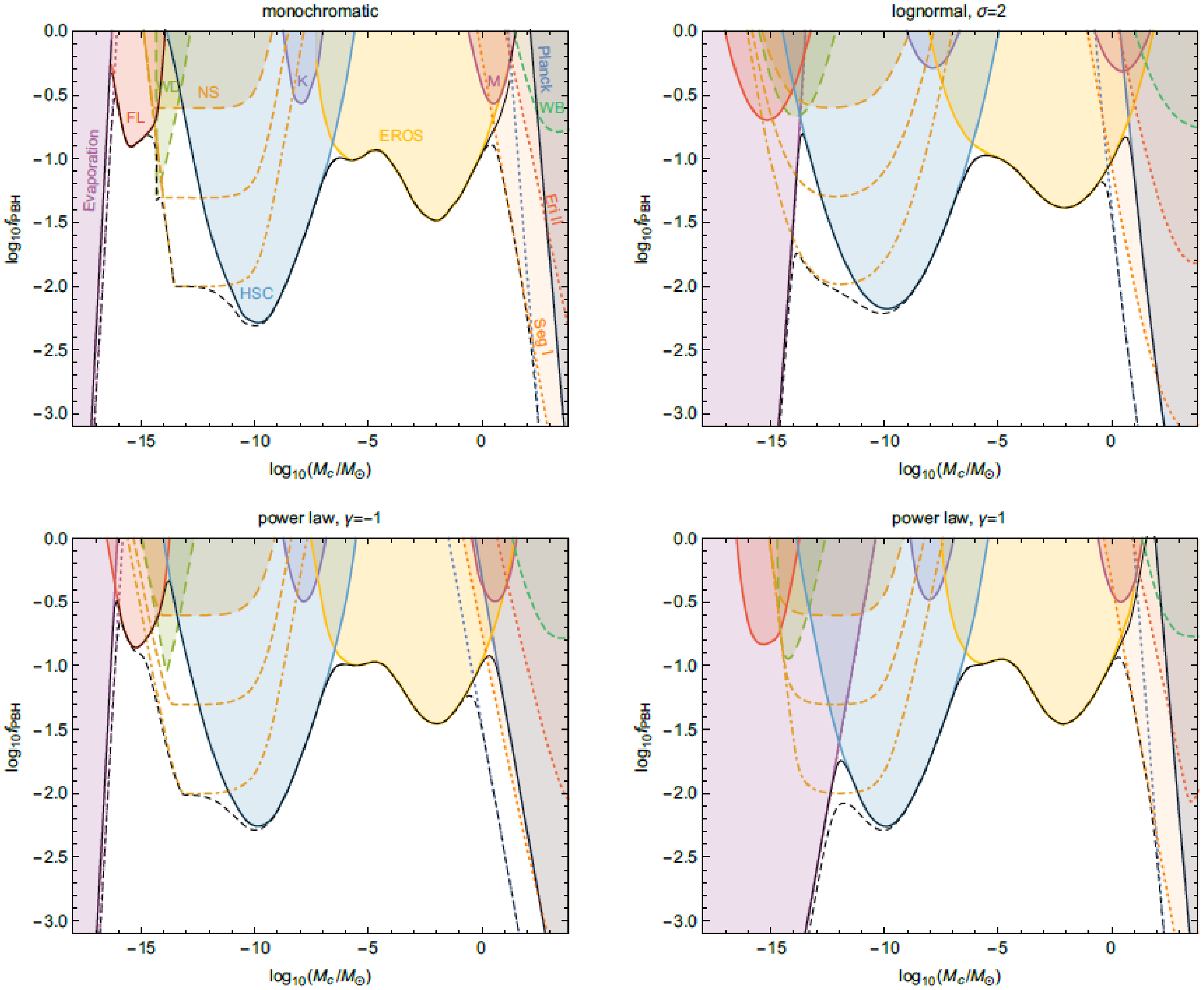
Mass functions always extended to some extent…





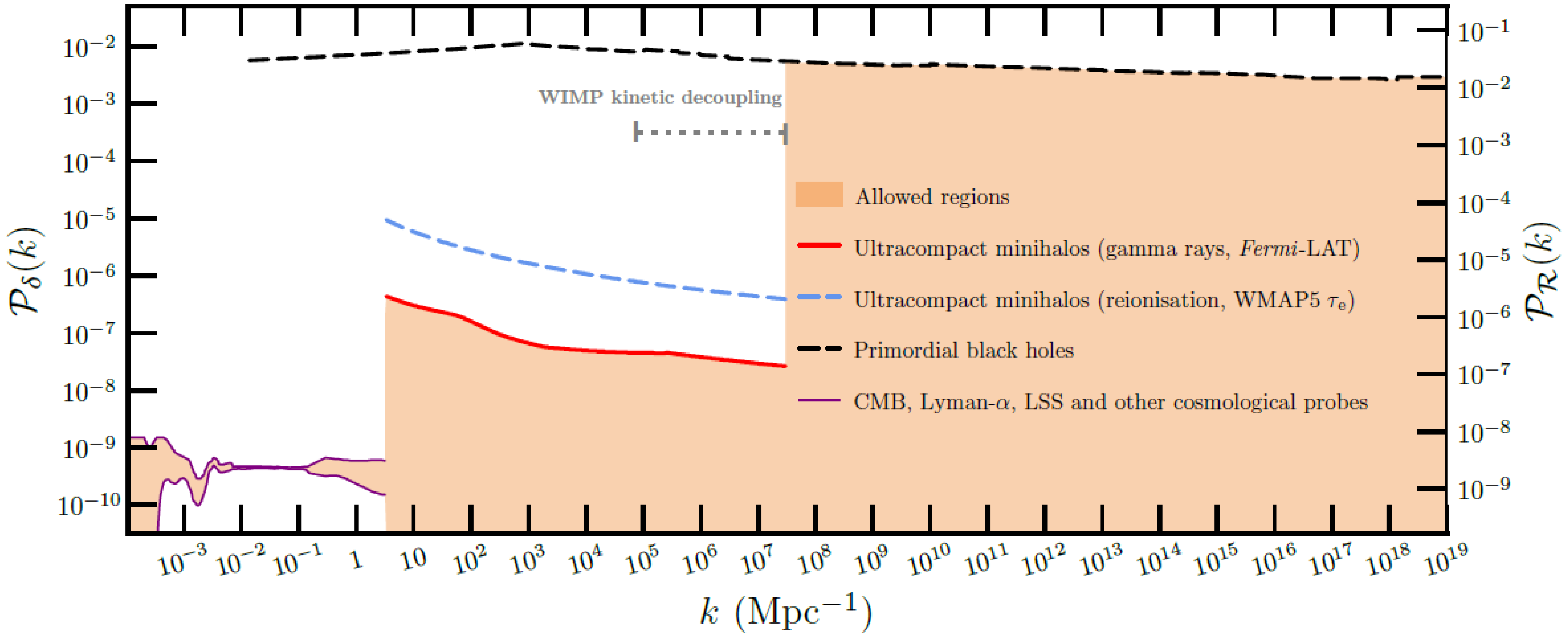
However the underlying power spectrum is hard to extract from the PBH mass function due to degeneracies between the effect of the amplitude and shape of the power spectrum… The PBH mass function would have to be observed with very high precision in order to reconstruct the shape of the primordial power spectrum near the corresponding peak. (Byrnes, Cole, Patil 2018)

Cosmological and astrophysical bounds on PBH abundances



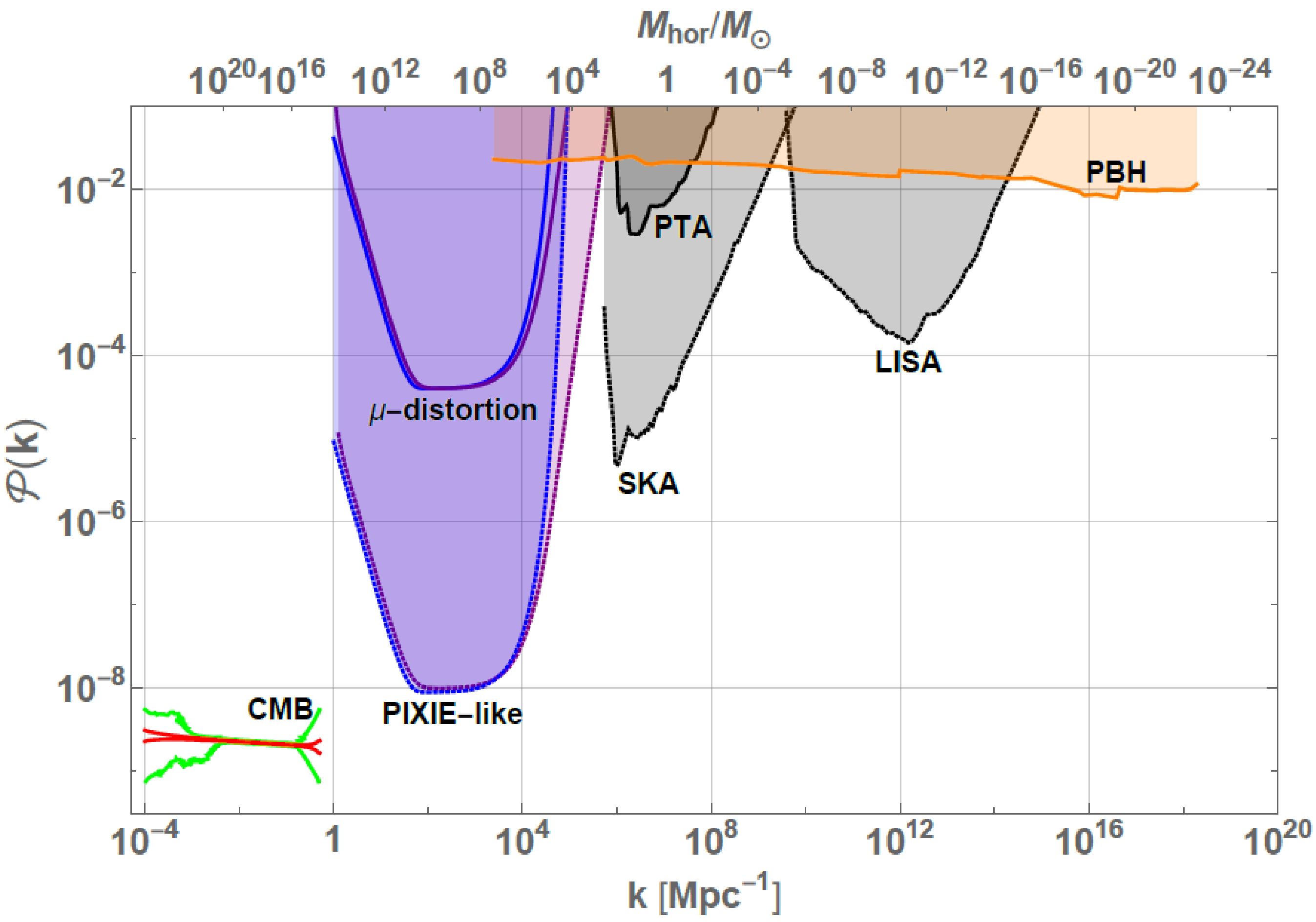
from B. Carr et al. 2017

Cosmological and astrophysical bounds on small scale power



from Bringmann, Scott, Akrami 2013

Shape dependence of constraints on underlying power spectrum



Byrnes, Cole, Patil (2018)

Primordial black holes from inflation?

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \qquad \epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2M_{\mathrm{pl}}^2},$$

$$\eta = \frac{\dot{\epsilon}}{\epsilon H}. \qquad \epsilon_{i+1} := \frac{\dot{\epsilon}_i}{H\epsilon_i}.$$

When potential is exactly flat, $dV/d\phi = 0$ so that $-\frac{\ddot{\phi}}{\dot{\phi}H} = \epsilon - \frac{\eta}{2} = 3.$

Therefore the smallest value for a monotonic potential is

$$\eta = -6 \rightarrow \epsilon \propto e^{-6N}$$

(Ultra Slow-Roll Inflation)

Primordial black holes from inflation?

$$\phi(\mathcal{N}) = \phi_* \pm M_{\rm pl} \int_{\mathcal{N}_*}^{\mathcal{N}} d\mathcal{N}' \sqrt{2\epsilon(\mathcal{N}')},$$

$$V(\mathcal{N}) = V(\mathcal{N}_*) \exp\left[-\frac{1}{3}\int_{\mathcal{N}_*}^{\mathcal{N}} d\mathcal{N}' \left(\frac{d\epsilon}{d\mathcal{N}'} + 6\epsilon\right)\right]$$

Given an arbitrary profile for $\epsilon(N)$, can reconstruct potential:

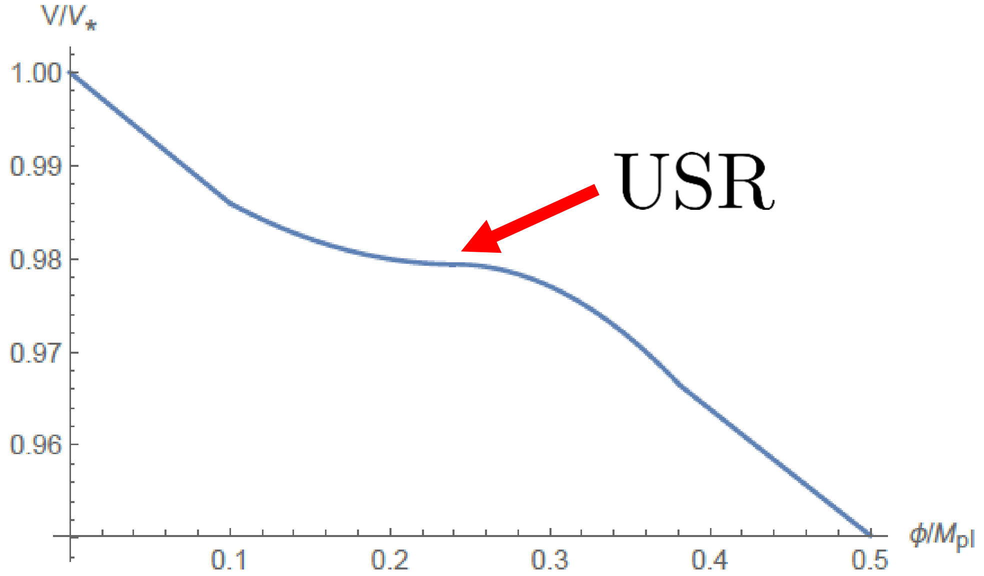
$$V_1(\phi) = V_* e^{-\frac{\sqrt{2\epsilon_0}}{M_{\rm pl}}(\phi-\phi_*)} \qquad \phi < \phi_1,\ \eta = 0$$

$$V_2(\phi) = V_1(\phi_1) e^{\frac{(\phi-\phi_1)^2}{3M_{\rm pl}^2} - \sqrt{2\epsilon_0}\frac{2(\phi-\phi_1)}{3M_{\rm pl}}} \qquad \phi_1 < \phi < \phi_2,\ \eta = -2$$

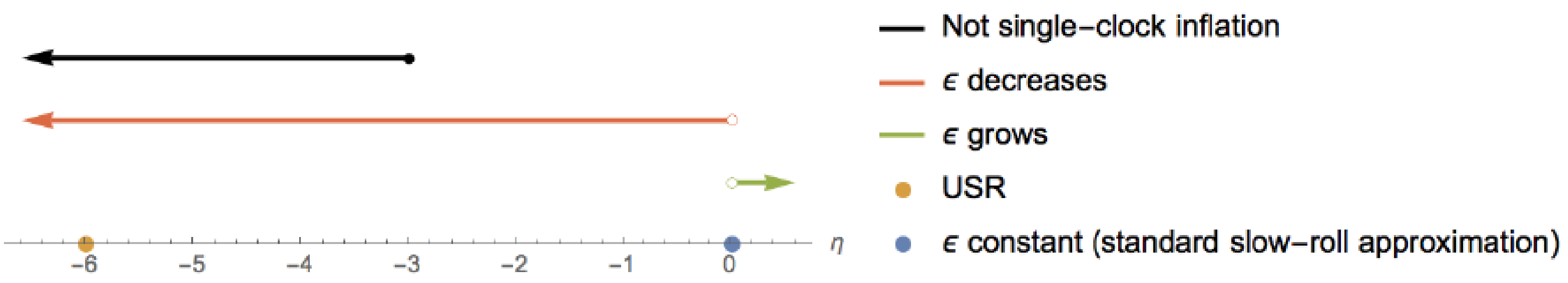
$$V_3(\phi) = V_2(\phi_2) = \text{constant} \qquad \phi_2 < \phi < \phi_3,\ \eta = -6$$

$$V_4(\phi) = V_3 e^{\frac{-2(\phi-\phi_3)^2}{3M_{\rm pl}^2} - \sqrt{2\epsilon_2}\frac{4(\phi-\phi_3)}{3M_{\rm pl}}} \qquad \phi_3 < \phi < \phi_4,\ \eta = +2$$

$$V_5(\phi) = V_4(\phi_4) e^{-\frac{\sqrt{2\epsilon_3}}{M_{\rm pl}}(\phi-\phi_4)} \qquad \phi_4 < \phi,\ \eta = 0$$



Primordial black holes from inflation?

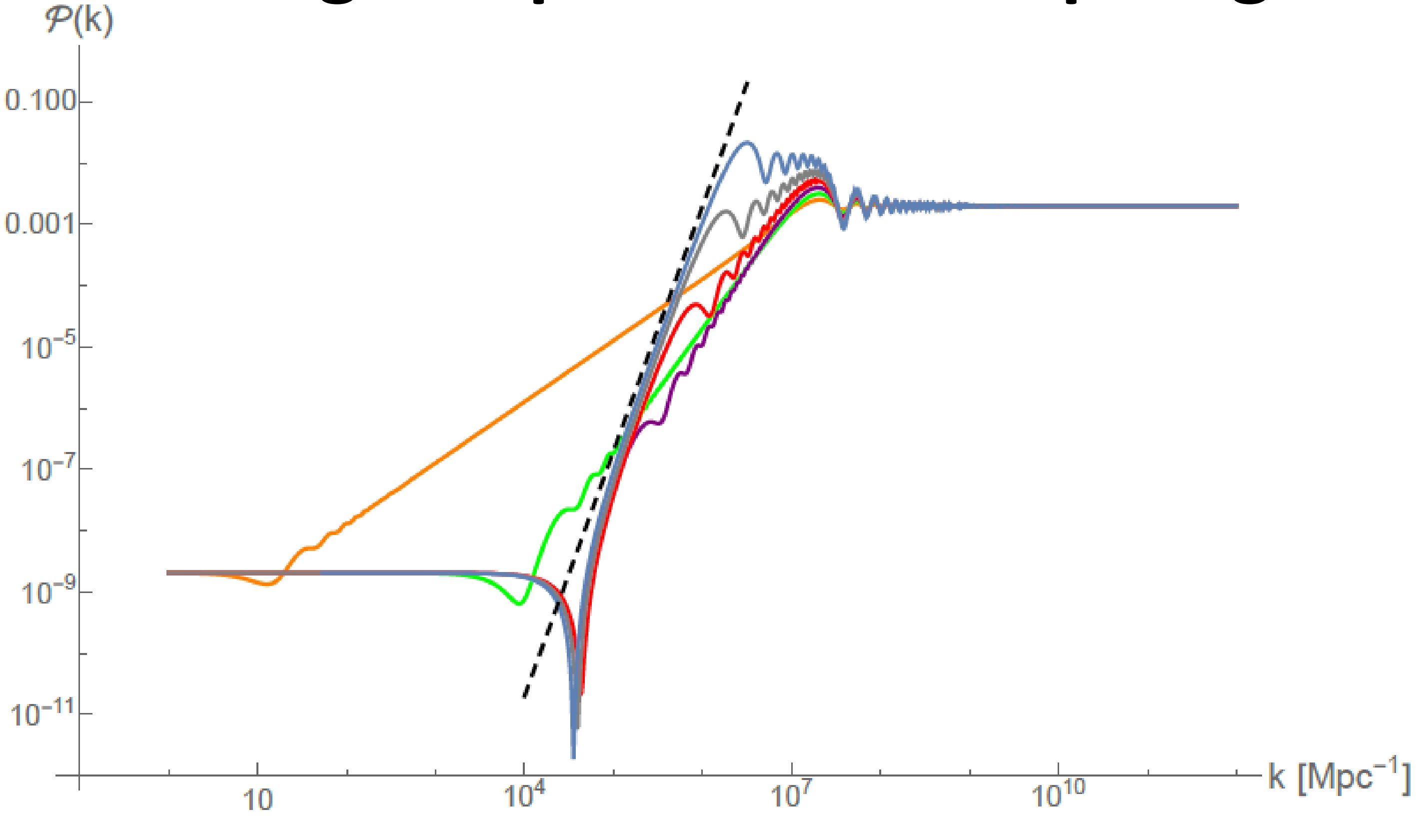


$$\mathcal{R}_k'' + 2\frac{z'}{z}\mathcal{R}_k' + k^2\mathcal{R}_k = 0 \qquad z^2 = 2a^2M_{\rm pl}^2\epsilon$$

$$\mathcal{R}_{k\to 0} = C_k + D_k\int^{\tau}\frac{d\tau'}{a^2\epsilon}.$$

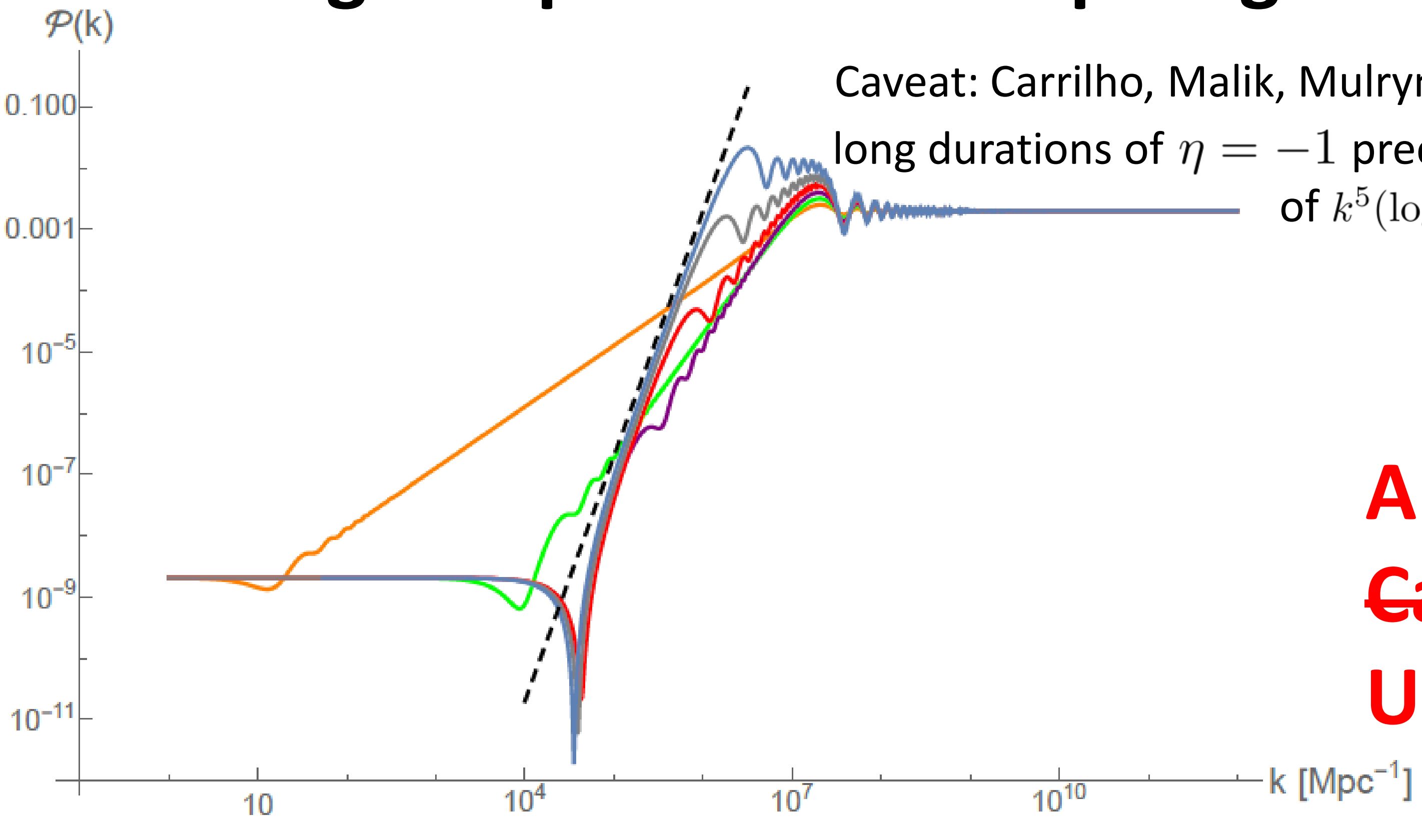
Formerly decaying mode can grow if $\eta < -3$

Analytic matching computation – steepest growth of $\propto k^4$



$$\frac{\mathcal{P}_{\mathcal{R}}(k)}{\mathcal{P}_{\mathcal{R}}(0)} = \left[1 - \frac{4}{5}\left(\frac{k}{k_{\mathrm{u}}}\right)^2 e^{3N_{\mathrm{USR}}}\right]^2 + 2\left(\frac{k}{k_{\mathrm{u}}}\right)^2 - 0.10\left(\frac{k}{k_{\mathrm{u}}}\right)^6 e^{6N_{\mathrm{USR}}} + 0.0075\left(\frac{k}{k_{\mathrm{u}}}\right)^8 e^{6N_{\mathrm{USR}}} + \cdots$$

Analytic matching computation – steepest growth of $\propto k^4$



Caveat: Carrilho, Malik, Mulryne 2019; if (unphysically) long durations of $\eta = -1$ preceding $\eta = -6$, growth of $k^5(\log k)^2$ possible…

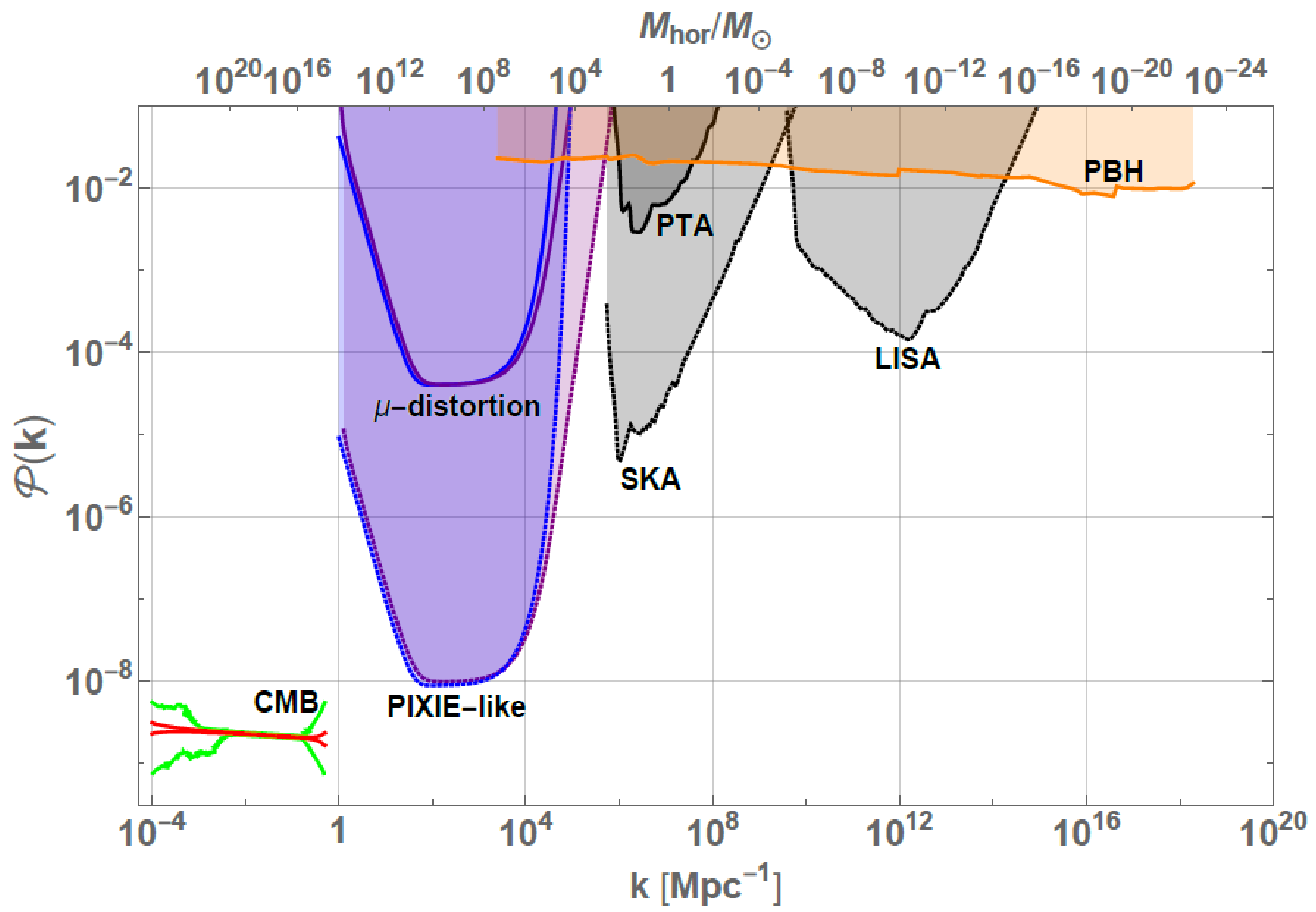
Analyticity?

~~Causality?~~

Unitarity?

$$\frac{\mathcal{P}_{\mathcal{R}}(k)}{\mathcal{P}_{\mathcal{R}}(0)} = \left[1 - \frac{4}{5}\left(\frac{k}{k_{\mathrm{u}}}\right)^2 e^{3N_{\mathrm{USR}}}\right]^2 + 2\left(\frac{k}{k_{\mathrm{u}}}\right)^2 - 0.10\left(\frac{k}{k_{\mathrm{u}}}\right)^6 e^{6N_{\mathrm{USR}}} + 0.0075\left(\frac{k}{k_{\mathrm{u}}}\right)^8 e^{6N_{\mathrm{USR}}} + \cdots$$

Can be used to extrapolate observational constraints at any given scale to much smaller scales…



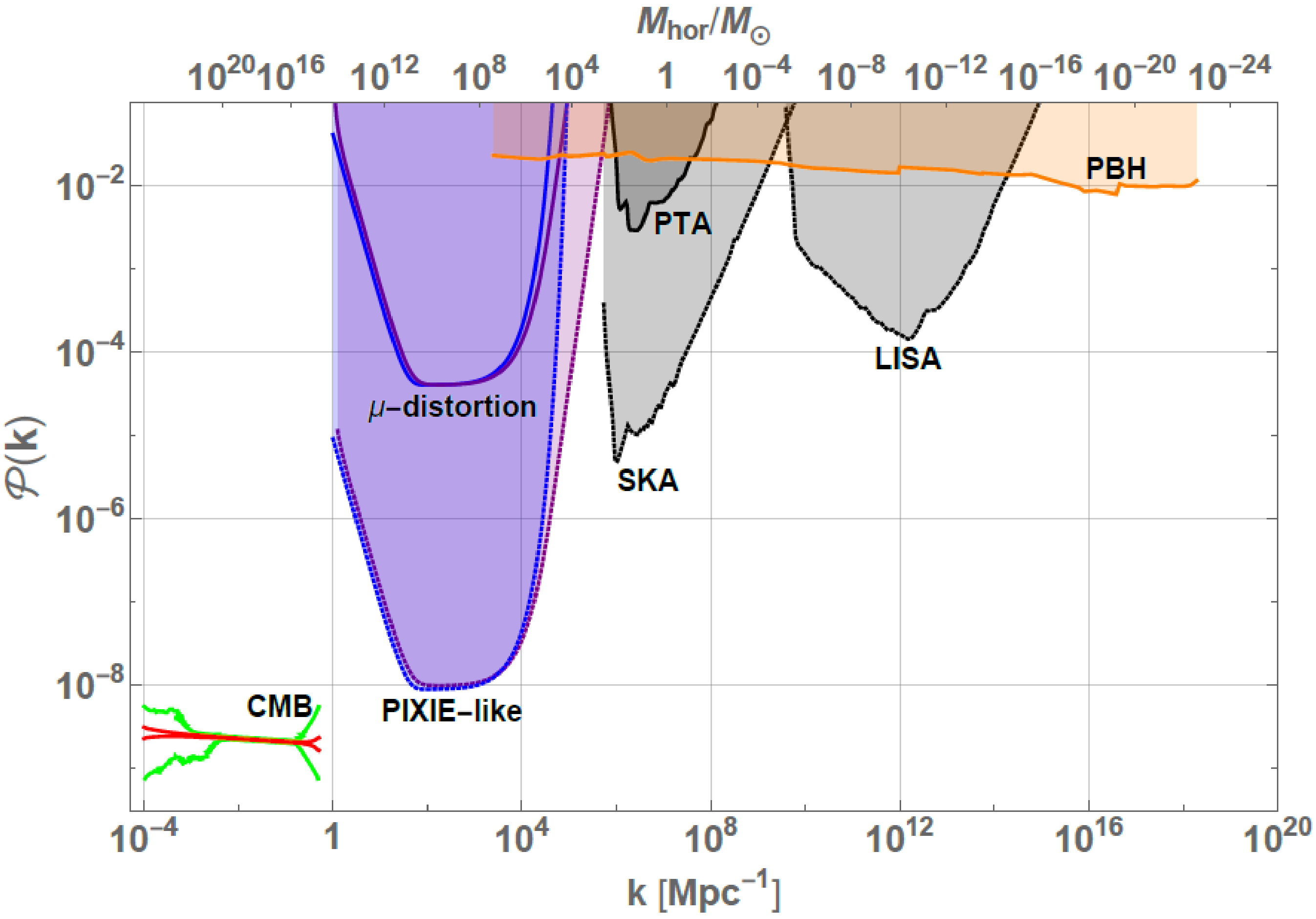
CMB spectral distortions

$$\mu_{\mathcal{R}} \approx \int_{k_{\rm min}}^{\infty} \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k)\, W_{\mathcal{R},\mu}(k),$$

$$W_{\mathcal{R},\mu}(k) \approx 2.27 \left[\exp\left(- \left[\frac{\hat{k}}{1360}\right]^2 \Big/ \left[1 + \left[\frac{\hat{k}}{260}\right]^{0.3} + \frac{\hat{k}}{340} \right] \right) - \exp\left(- \left[\frac{\hat{k}}{32}\right]^2 \right) \right],$$

with $\hat{k} = k/[1\,\mathrm{Mpc}^{-1}]$ and $k_{\rm min} \simeq 1\,\mathrm{Mpc}^{-1}$

Can be used to extrapolate observational constraints at any given scale to much smaller scales…



Gravitational waves

$$\mathcal{L}_{\text{int}} = h_{ij}\partial_i\mathcal{R}\partial_j\mathcal{R}$$

$$\mathcal{P}_h(\tau, k) = \int_0^\infty dv \int_{|1-v|}^{1+v} du\, K(\tau, u, v)\mathcal{P}_{\mathcal{R}}(ku)\mathcal{P}_{\mathcal{R}}(kv)$$

with theoretical constraints, FIRAS bounds extend to close gap on lowest mass for Primordial SMBH seeds…

Many > O(1) theoretical uncertainties remain!

- Range in critical density (e.g. during RD) $\Delta_c \sim 0.3 \to 0.66$ from different assumptions e.g. for radial profiles;
- Non-sphericity of collapse; effects of vorticity during MD;
- Non-linear structure formation formalism (standard Press-Schecter can be improved). Choice of window function when calculating variance – top hat, or Gaussian?
- **Non-Gaussianities!** Especially the varieties that are not captured by templates used to constrain presence in CMB.
- Can we improve observational constraints on small scale power to exclude certain mass ranges?

Lots of theorists still puzzling over all of this!

Thanks for your attention!

Cf. spectral dependence of sensitivity curves for interferometry; Thrane, Romano (2013)

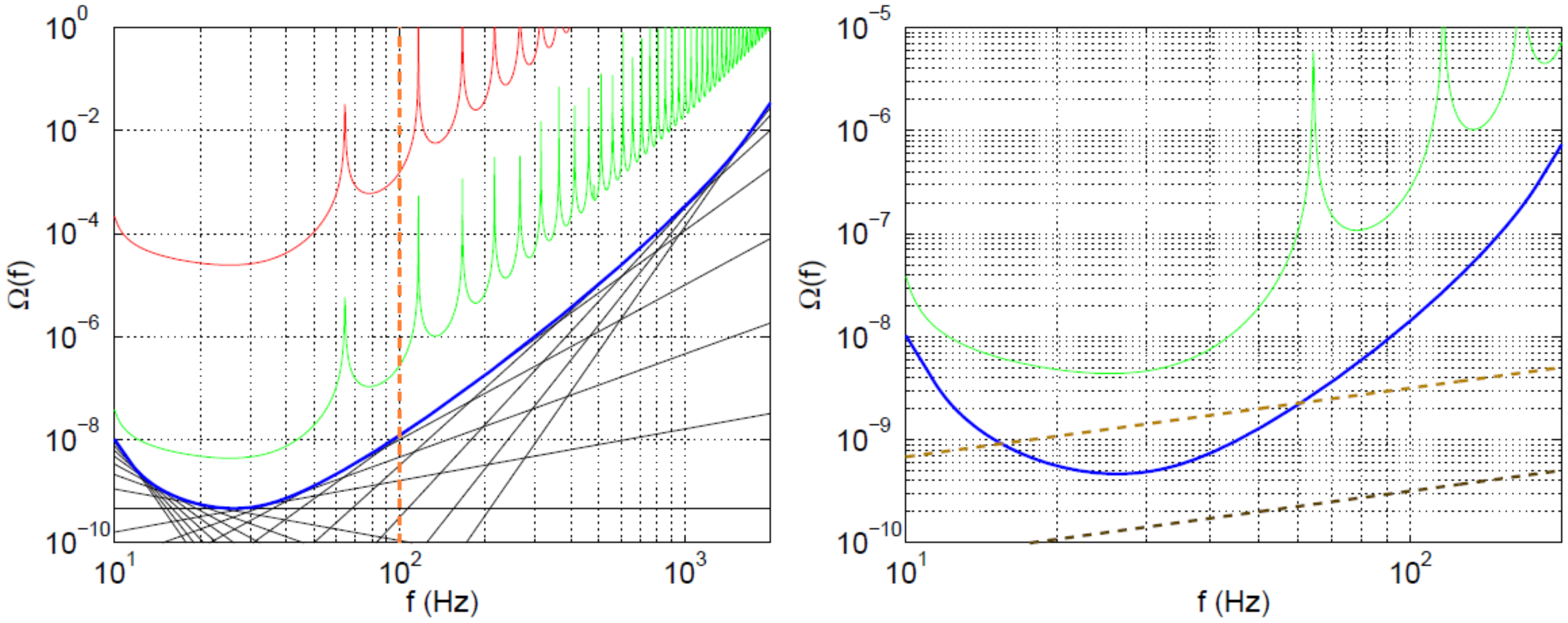


FIG. 7: Left panel: $\Omega_{\rm gw}(f)$ sensitivity curves from different stages in a potential future Advanced LIGO H1L1 correlation search for power-law gravitational-wave backgrounds. The red line shows the effective strain spectral density $S_{\rm eff}(f) = P_n(f)/|\Gamma_{\rm H1L1}(f)|$ of the H1L1 detector pair to a gravitational-wave background signal converted to energy density $\Omega_{\rm eff}(f)$ via Eq. 3. (The $P_n(f)$ used in this calculation is the design detector noise power spectral density for an Advanced LIGO detector, assumed to be the same for both H1 and L1.) The spikes in the red curve are due to zeroes in the overlap reduction function $\Gamma_{\rm H1L1}(f)$, which is shown in the left panel of Fig. 5. The green curve, $S_{\rm eff}(f)/\sqrt{2T\delta f}$, is obtained through the optimal combination of one year's worth of data, assuming a frequency bin width of 0.25 Hz as is typical [2]. The vertical dashed orange line marks a typical Advanced LIGO reference frequency, $f_{\rm ref} = 100\,{\rm Hz}$. The set of black lines are obtained by performing the integration in Eq. 29 for different power law indices β, requiring that $\rho = 1$ to determine Ω_β. Finally, the blue power-law integrated sensitivity curve is the envelope of the black lines. Right panel: a demonstration of how to interpret a power-law integrated curve. The thin green line and thick blue line are the same as in the left panel. The two dashed