Modular Invariance Approach to the Flavour Problem

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Understanding the origin of the pattern of neutrino mixing and of neutrino mass squared differences that emerged from the neutrino oscillation data in the recent years is one of the most challenging problems in neutrino physics. It is part of the more general fundamental problem in particle physics of understanding the origins of flavour in the quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing.

"Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."

From Model Physicist, CERN Courier, 13 October 2017.

Of fundamental importance are also

- the determination of the status of lepton charge conservation and the nature - Dirac or Majorana - of massive neutrinos (which is one of the most challenging and pressing problems in present day elementary particle physics) (GERDA, CUORE, KamLAND-Zen, EXO, LEGEND, nEXO,...);

- determining the status of CP symmetry in the lepton sector (T2K, NOνA; T2HK, DUNE);

- determination of the type of spectrum neutrino masses possess, or the "neutrino mass ordering" (T2K + NOνA; JUNO; PINGU, ORCA; T2HKK, DUNE);

- determination of the absolute neutrino mass scale, or $\min(m_j)$ (KATRIN, new ideas; cosmology).

The program of research extends beyond 2030.

- **BS3νRM: eV scale sterile ν's; NSI's; ChLFV processes ($\mu \to e + \gamma$, $\mu \to 3e$, $\mu^- - e^-$ conversion on (A,Z)); $\nu-$related BSM physics at the TeV scale (N_{jR}, H^{--}, H^-, etc.).**

Reference Model: 3-ν mixing

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \qquad l = e, \mu, \tau.$$

The PMNS matrix U - 3×3 unitary.
ν_j, $m_j \neq 0$: Dirac or Majorana particles.

Data: 3 νs are light: $\nu_{1,2,3}$, $m_{1,2,3} \lesssim 0.5$ eV.

3-ν mixing: 3-flavour neutrino oscillations possible.

ν_μ, E; at distance L: $P(\nu_\mu \rightarrow \nu_{\tau(e)}) \neq 0$, $P(\nu_\mu \rightarrow \nu_\mu) < 1$

$P(\nu_l \rightarrow \nu_{l'}) = P(\nu_l \rightarrow \nu_{l'}; E, L; U; m_2^2 - m_1^2, m_3^2 - m_1^2)$

Three Neutrino Mixing

$$\nu_{l\mathrm{L}} = \sum_{j=1}^{3} U_{lj}\, \nu_{j\mathrm{L}} \ .$$

U is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix,

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

- U - $n \times n$ unitary:

	n	2	3	4
mixing angles:	$\frac{1}{2}n(n-1)$	1	3	6
CP-violating phases:				
• ν_j- Dirac:	$\frac{1}{2}(n-1)(n-2)$	0	1	3
• ν_j- Majorana:	$\frac{1}{2}n(n-1)$	1	3	6

$n = 3$: 1 Dirac and
2 additional CP-violating phases, Majorana phases

S.M. Bilenky, J. Hosek, S.T.P., 1980

PMNS Matrix: Standard Parametrization

$$U = V\,P\,, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix},$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

- $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, $\theta_{ij} = [0, \frac{\pi}{2}]$,
- δ - Dirac CPV phase, $\delta = [0, 2\pi]$; CP inv.: $\delta = 0, \pi, 2\pi$;
- α_{21}, α_{31} - Majorana CPV phases; CP inv.: $\alpha_{21(31)} = k(k')\pi$, $k(k') = 0, 1, 2...$

S.M. Bilenky et al., 1980

- $\Delta m^2_{\odot} \equiv \Delta m^2_{21} \cong 7.37 \times 10^{-5}$ eV$^2 > 0$, $\sin^2\theta_{12} \cong 0.297$, $\cos 2\theta_{12} \gtrsim 0.29$ (3σ),
- $|\Delta m^2_{31(32)}| \cong 2.53$ (2.43) [2.56 (2.54)] $\times 10^{-3}$ eV^2, $\sin^2\theta_{23} \cong$ 0.437 (0.569) [0.425 (0.589)], NO (IO) ,
- θ_{13} - the CHOOZ angle: $\sin^2\theta_{13} = 0.0214$ (0.0218) [0.0215 (0.0216)], NO (IO).

F. Capozzi et al. (Bari Group), arXiv:1601.07777 [arXiv:1703.04471].

- $\text{sgn}(\Delta m^2_{\text{atm}}) = \text{sgn}(\Delta m^2_{31(32)})$ not determined

$$\Delta m^2_{\text{atm}} \equiv \Delta m^2_{31} > 0, \quad \text{normal mass ordering (NO)}$$

$$\Delta m^2_{\text{atm}} \equiv \Delta m^2_{32} < 0, \quad \text{inverted mass ordering (IO)}$$

Convention: $m_1 < m_2 < m_3$ - NO, $m_3 < m_1 < m_2$ - IO

$$\Delta m^2_{31}(NO) = -\,\Delta m^2_{32}(IO)$$

$$m_1 \ll m_2 < m_3, \qquad \text{NH},$$

$$m_3 \ll m_1 < m_2, \qquad \text{IH},$$

$$m_1 \cong m_2 \cong m_3,\; m^2_{1,2,3} >> |\Delta m^2_{31(32)}|,\; \text{QD};\; m_j \gtrsim 0.10\ \text{eV}.$$

- $m_2 = \sqrt{m_1^2 + \Delta m^2_{21}}$, $m_3 = \sqrt{m_1^2 + \Delta m^2_{31}}$ - NO;
- $m_1 = \sqrt{m_3^2 + \Delta m^2_{23} - \Delta m^2_{21}}$, $m_2 = \sqrt{m_3^2 + \Delta m^2_{23}}$ - IO;

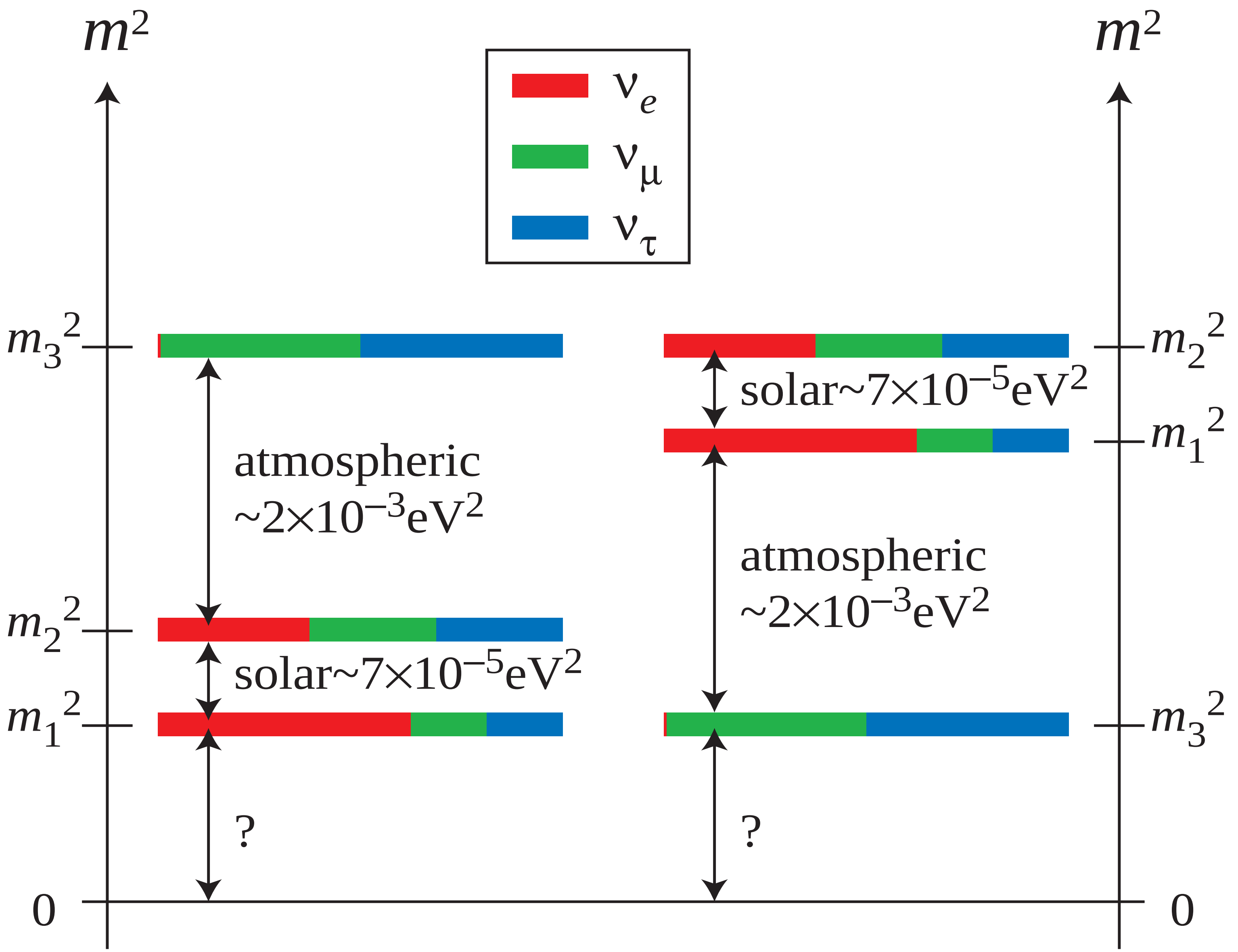

m^2
m^2
ν_e
ν_μ
ν_τ
$m_3^{\ 2}$
atmospheric
$\sim 2\times 10^{-3}$eV^2
$m_2^{\ 2}$
solar$\sim 7\times 10^{-5}$eV^2
$m_1^{\ 2}$
?
0
$m_2^{\ 2}$
solar$\sim 7\times 10^{-5}$eV^2
$m_1^{\ 2}$
atmospheric
$\sim 2\times 10^{-3}$eV^2
$m_3^{\ 2}$
?
0

Table 3: Best fit values and allowed ranges at $N\sigma = 1, 2, 3$ for the 3ν oscillation parameters, in either NO or IO. The latter column shows the formal "1σ accuracy" for each parameter, defined as 1/6 of the 3σ range divided by the best-fit value (in percent).

Parameter	Ordering	Best fit	1σ range	2σ range	3σ range	"1σ" (%)
$\Delta m^2_{\odot}/10^{-5}$ eV^2	NO	7.34	7.20 – 7.51	7.05 – 7.69	6.92 – 7.91	2.2
	IO	7.34	7.20 – 7.51	7.05 – 7.69	6.92 – 7.91	2.2
$\lvert\Delta m^2_{\mathrm{A}}\rvert/10^{-3}$ eV^2	NO	2.49	2.46 – 2.53	2.43 – 2.56	2.39 – 2.59	1.4
	IO	2.48	2.44 – 2.51	2.41 – 2.54	2.38 – 2.58	1.4
$\sin^2\theta_{12}$	NO	3.04	2.91 – 3.18	2.78 – 3.32	2.65 – 3.46	4.4
	IO	3.03	2.90 – 3.17	2.77 – 3.31	2.64 – 3.45	4.4
$\sin^2\theta_{13}/10^{-2}$	NO	2.14	2.07 – 2.23	1.98 – 2.31	1.90 – 2.39	3.8
	IO	2.18	2.11 – 2.26	2.02 – 2.35	1.95 – 2.43	3.7
$\sin^2\theta_{23}/10^{-1}$	NO	5.51	4.81 – 5.70	4.48 – 5.88	4.30 – 6.02	5.2
	IO	5.57	5.33 – 5.74	4.86 – 5.89	4.44 – 6.03	4.8
δ/π	NO	1.32	1.14 – 1.55	0.98 – 1.79	0.83 – 1.99	14.6
	IO	1.52	1.37 – 1.66	1.22 – 1.79	1.07 – 1.92	9.3

$$\Delta m^2_{\odot} \equiv \Delta m^2_{21}; \quad \Delta m^2_{\mathrm{A}} \equiv \Delta m^2_{31(32)}, \text{ NO (IO)}.$$

F. Capozzi et al. (Bari Group), arXiv:1804.09678.

- Dirac phase δ: $\nu_l \leftrightarrow \nu_{l'}$, $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$, $l \neq l'$; $A^{(l,l')}_{\rm CP} \propto J_{\rm CP} \propto \sin\theta_{13} \sin\delta$:

P.I. Krastev, S.T.P., 1988

$$J_{CP} = {\rm Im}\left\{U_{e1}\, U_{\mu 2}\, U^*_{e2}\, U^*_{\mu 1}\right\} = \frac{1}{8}\sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta$$

Current data: $|J_{CP}| \lesssim 0.035$ (can be relatively large!); b.f.v. with $\delta = 3\pi/2$: $J_{CP} \cong -0.035$.

- Majorana phases α_{21}, α_{31}:
  - $\nu_l \leftrightarrow \nu_{l'}$, $\bar{\nu}_l \leftrightarrow \bar{\nu}_{l'}$ not sensitive;

S.M. Bilenky et al., 1980;
P. Langacker et al., 1987

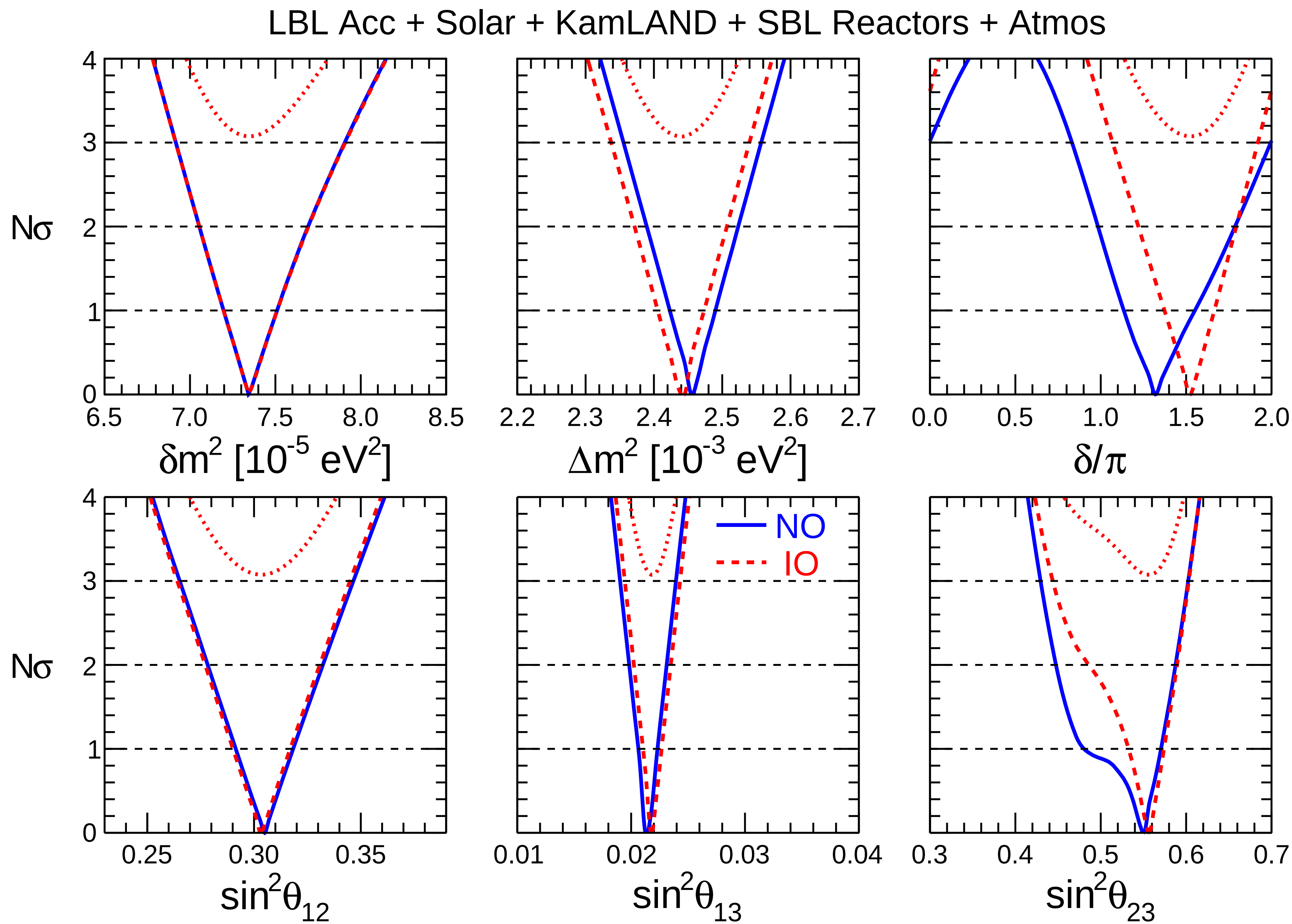
  - $|<m>|$ in $(\beta\beta)_{0\nu}$–decay depends on α_{21}, α_{31};
  - $\Gamma(\mu \rightarrow e + \gamma)$ etc. in SUSY theories depend on $\alpha_{21,31}$;
  - BAU, leptogenesis scenario: $\delta, \alpha_{21,31}$!

$$\delta \cong 3\pi/2?$$

$$\begin{aligned} J_{CP} &= \mathrm{Im}\left\{U_{e1}\, U_{\mu 2}\, U^*_{e2}\, U^*_{\mu 1}\right\} \\ &= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \sin\delta \end{aligned}$$

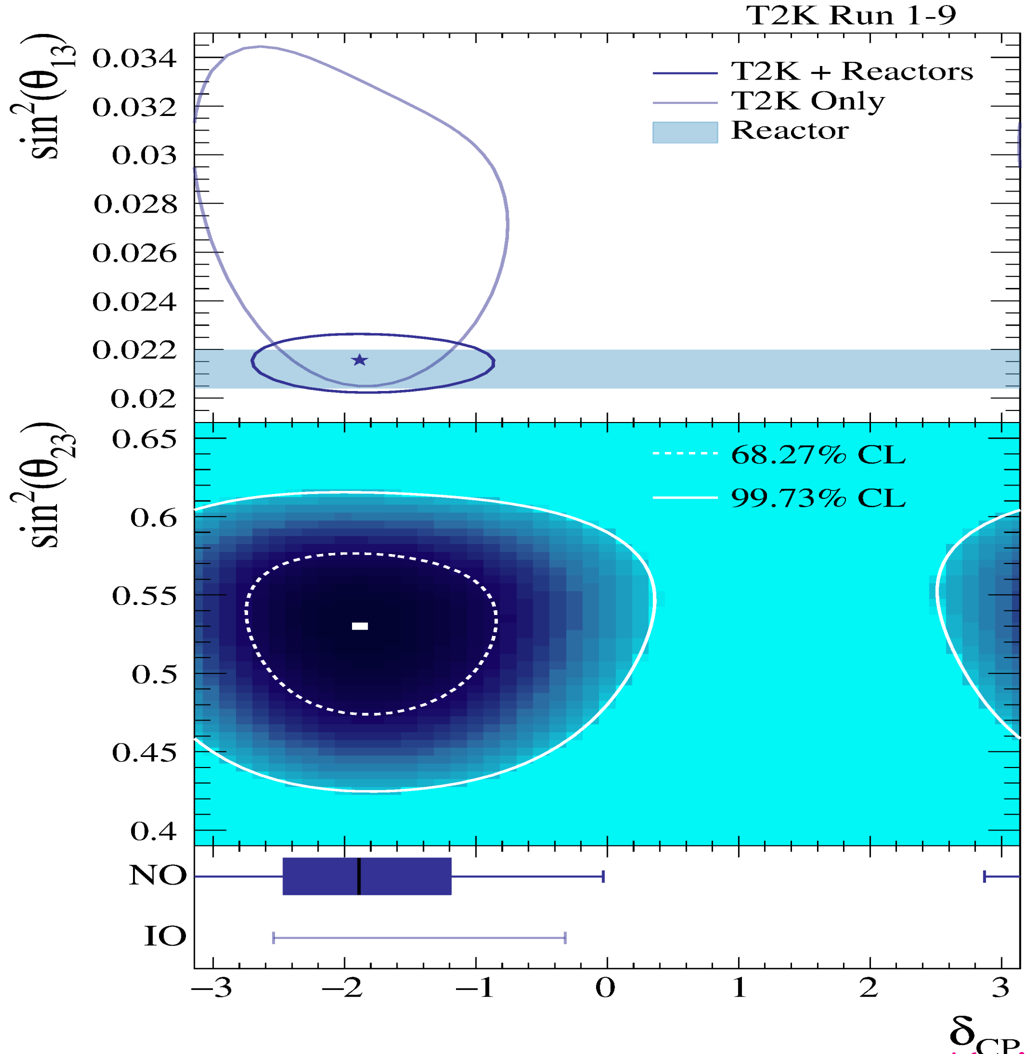
- **Best fit value:** $\delta = 1.32\,(1.52)\pi$ **[**$1.30\,(1.54)\pi$**];**
- $\delta = 0$ **or** 2π **are disfavored at** $3.0\,(3.6)\sigma$ **[**$2.6\,(3.0)\sigma$**];**
- $\delta = \pi$ **is disfavored at** $1.8\,(3.6)\sigma$ **[**$1.7\,(3.3)\sigma$**];**
- $\delta = \pi/2$ **is strongly disfavored at** $4.4\,(5.2)\sigma$ **[**$4.3\,(5.0)\sigma$**].**
- **At** 3σ**:** δ/π **is found to lie in 0.83-1.99 (1.07-1.92) [1.07-1.97 (0.80-2.08)].**

F. Capozzi, E. Lisi *et al.*, arXiv:1804.09678 [E. Esteban *et al.*, NuFit 3.2 (Jan., 2018)]



F. Capozzi et al. (Bari Group), arXiv:1804.09678.

Latest results from T2K



K. Abe et al., 1910.03887

Best fit value: $\delta = -1.89\,(-1.38)$, NO (IO).
$\delta = 0, \pi$ ruled out at 95% CL.
At 3σ: δ is found to lie in [-3.41,-0.03] ([-2.54,-0.32]), NO (IO).

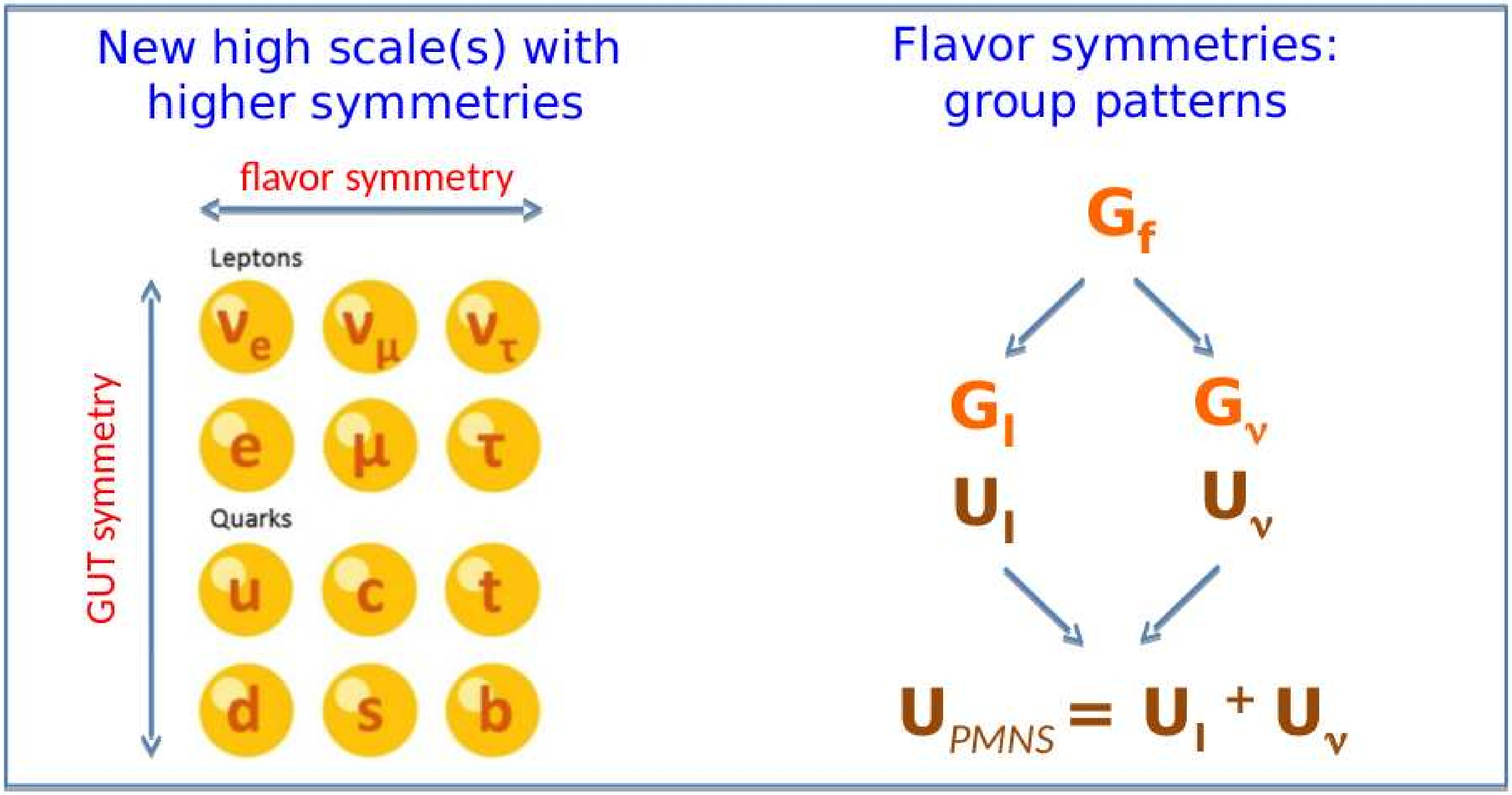
Latest global analysis: data favors NO

IO disfavored at 3.1σ.

F. Capozzi et al., 1804.09678.

The Flavour Problem

Model building with symmetries



E. Lisi, TAUP 2019

Modular Invariance

Modular invariance has been investigated in the context of field and superstring theories, being a feature of a number of theoretical physics constructions (theories with extra dimensions compactified on a torus (or tori), superstring theories on tori or orbifolds, supergravity theories) [2]-[7]; it can be present in theories with global or local super-symmetry and appears to be a property of the quantum Hall effect [8]-[13]. The modular forms which are an integral part of the approach (see below) have been extensively studied by mathematicians, in particular, in connection with number theory [14].

[2] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, Phys. Rept. 445, 1 (2007). [3] L. E. Ibanez, Phys. Lett. B181, 269 (1986). [4] S. Hamidi and C. Vafa, Nucl. Phys. B279, 465 (1987). [5] S. Ferrara, D. Lust and S. Theisen, Phys. Lett. B233, 147 (1989). [6] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0405, 079 (2004). [7] S. Ferrara, D. Lust, A. D. Shapere and S. Theisen, Phys. Lett. B225, 363 (1989). [8] C. A. Ltken and G. G. Ross, Phys. Rev. D45, 11837 (1992). [9] A. Cappelli and G. R. Zemba, Nucl. Phys. B490, 595 (1997). [10] C. P. Burgess and B. P. Dolan, Phys. Rev. B63, 155309 (2001). [11] M. Lippert, R. Meyer and A. Taliotis, JHEP 1501, 023 (2015). [12] C.A. Lutken, EPJ Web Conf. 71, 0079 (2014) 00079 (doi:10.1051/epjconf/20147100079). [13] C. A. Lutken, Phys. Rev. B99, 195152 (2019). [14] H. M. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group, Graduate Studies in Mathematics, vol. 37, American Mathematical Society (2001).

The Modular Group and the Finite Modular Groups

The modular group $\overline{\Gamma}$ – group of linear fractional transformations γ acting on the complex variable τ belonging to the upper-half complex plane:

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}\,, \qquad a, b, c, d \in Z\,, \qquad ad - bc = 1\,,\; \mathrm{Im}\tau > 0\,.$$

$\overline{\Gamma}$ is generated by two transformations S and T satisfying

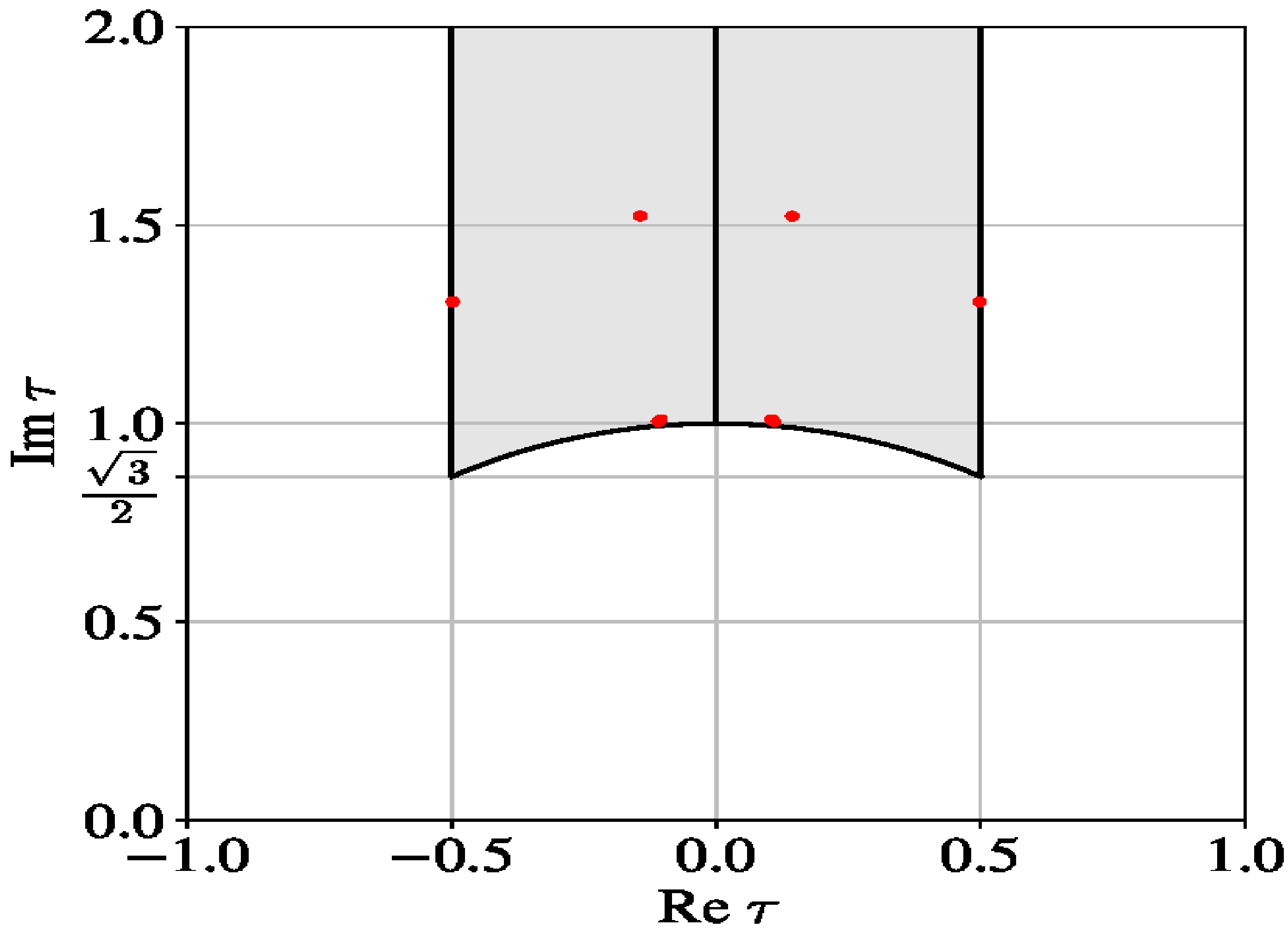
$$S^2 = (ST)^3 = I\,,$$

I being the identity element, and acting on τ as

$$\tau \xrightarrow{S} -\frac{1}{\tau}\,, \qquad \tau \xrightarrow{T} \tau + 1\,.$$

S and T can be represented as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$



The Fundamental Domain of $\overline{\Gamma}$ shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

$\overline{\Gamma}$ **is isomorphic to the** *projective special linear group* $PSL(2,Z) = SL(2,Z)/Z_2$**,** $SL(2,Z)$ **is the special linear group of** 2×2 **matrices with integer elements and unit determinant, and** $Z_2 = \{I, -I\}$ **is its centre.**
$SL(2,Z) = \Gamma(1) \equiv \Gamma$ **contains a** *series of infinite normal subgroups* $\Gamma(N)$**,**

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}\,,\ N = 1, 2, 3, \ldots\,,$$

called the *principal congruence subgroups***. For** $N = 1$ **and** 2**, we define the groups** $\overline{\Gamma}(N) \equiv \Gamma(N)/\{I, -I\}$ **with** $\overline{\Gamma}(1) \equiv \overline{\Gamma}$**. For** $N > 2$**,** $\overline{\Gamma}(N) \equiv \Gamma(N)$ **since** $\Gamma(N)$ **does not contain the subgroup** $\{I, -I\}$**.**

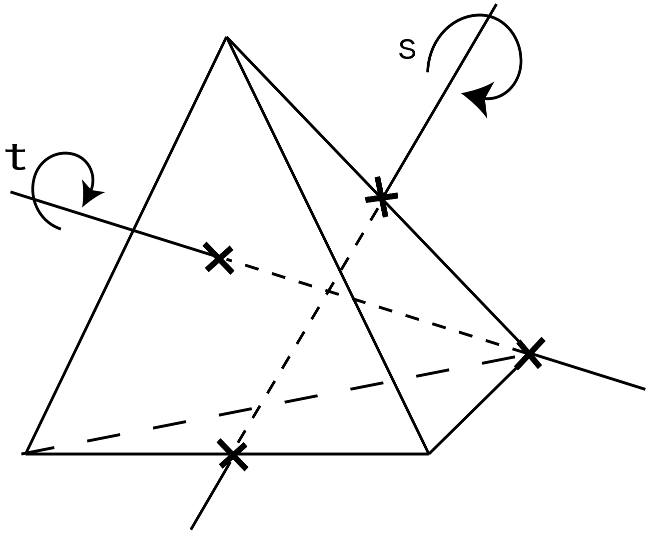
The *quotient groups* $\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$ *are called finite modular groups***. Remarkably, for** $N \leq 5$**,** Γ_N **are isomorphic to non-Abelian discrete groups widely used in flavour model building:**
$\Gamma_2 \simeq S_3$**,** $\Gamma_3 \simeq A_4$**,** $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$**.**
Γ_N **is presented by two generators** S **and** T **satisfying:**

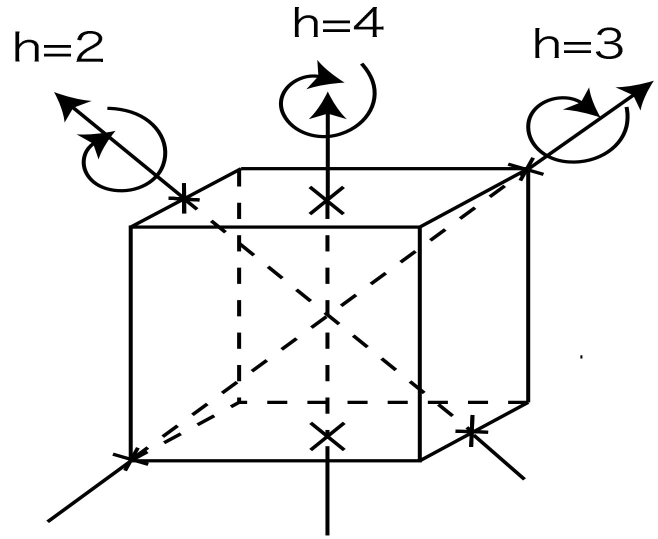
$$S^2 = (ST)^3 = T^N = I\,.$$

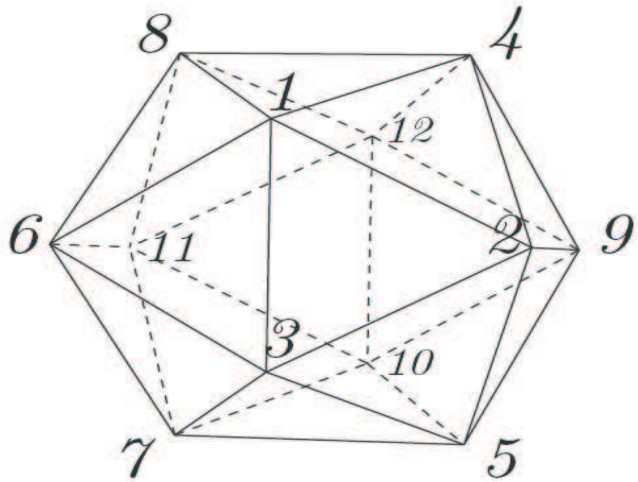
The group theory of $\Gamma_2 \simeq S_3$**,** $\Gamma_3 \simeq A_4$**,** $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$ **is summarised, e.g., in P.P. Novichkov** *et al.***, JHEP 07 (2019) 165, arXiv:1905.11970.**

Group	Number of elements	Generators	Irreducible representations
S_4	24	S, T (U)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{3'}$
A_4	12	S, T	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{3}$
T'	24	S, T (R)	$\mathbf{1}$, $\mathbf{1'}$, $\mathbf{1''}$, $\mathbf{2}$, $\mathbf{2'}$, $\mathbf{2''}$, $\mathbf{3}$
A_5	60	$\tilde{S}$, $\tilde{T}$	$\mathbf{1}$, $\mathbf{3}$, $\mathbf{3'}$, $\mathbf{4}$, $\mathbf{5}$

Number of elements, generators and irreducible representations of S_4, A_4, T' and A_5 discrete groups.







Examples of symmetries: A_4, S_4, A_5.

From M. Tanimoto et al., arXiv:1003.3552

Modular Forms

The key elements of the considered framework are modular forms $f(\tau)$ of *weight k and level N* **– holomorphic functions of τ, which transform under $\overline{\Gamma}(N)$ as follows:**

$$f\,(\gamma\tau) = (c\tau + d)^k\, f(\tau)\,, \qquad \gamma \in \overline{\Gamma}(N)\,,$$

where the weight $k \geq 0$ is an even number, level N is a natural number.
For given k, N, the modular forms span a linear space of finite dimension.
The dimension of the linear space of modular forms
of weight k and level 3, $\mathcal{M}_k(\Gamma_3 \simeq A_4)$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4 \simeq S_4)$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5 \simeq A_5)$, is $5k+1$.
Thus, $\dim\mathcal{M}_2(\Gamma_3 \simeq A_4) = 3$, $\dim\mathcal{M}_2(\Gamma_4 \simeq S_4) = 5$, $\dim\mathcal{M}_2(\Gamma_5 \simeq A_5) = 11$.
One can find a basis $F(\tau) \equiv (f_1(\tau), f_2(\tau), \dots)^T$ in each of these spaces such that for any $\gamma \in \overline{\Gamma}$, $F(\gamma\tau)$ belongs to the same space and transforms according to a unitary irreducible representation $\mathbf{r}$ of Γ_N:

$$F\,(\gamma\tau) = (c\tau + d)^k\, \rho_{\mathbf{r}}(\gamma) F(\tau)\,, \qquad \gamma \in \overline{\Gamma}\,.$$

This result is at the basis of the modular invariance approach to the flavour problem proposed in F. Feruglio, arXiv:1706.08749.

It is of crucial importance to find the basis of modular forms of weight 2 transforming in irreps of Γ_N.
Multiplets of Γ_N of higher weight modular forms can be constructed from tensor products of the lowest weight 2 multiplets (they represent homogenious polynomials of the weight 2 modular forms).

For ($\Gamma_3 \simeq A_4$), the generating (basis) modular forms of weight 2 were shown to form a 3 of A_4 (expressed in terms of the Dedekin eta function).

F. Feruglio, arXiv:1706.08749

For ($\Gamma_4 \simeq S_4$), the 5 basis modular forms of weight 2 were shown to form a 2 and a $3'$ of S_4 (expressed in terms of the Dedekin eta function).

J. Penedo, STP, arXiv:1806.11040

For ($\Gamma_5 \simeq A_5$), the 11 basis modular forms of weight 2 were shown to form a 3, a $3'$ and a 5 of A_5 (expressed in terms of the Jacobi theta function).

P.P. Novichkov, J. Penedo, STP, A.V. Titov, arXiv:1812.02158

For ($\Gamma_2 \simeq S_3$), the 2 basis modular forms of weight 2 were shown to form a 2 of S_3 (expressed in terms of the Dedekin eta function).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N = 4$ (i.e., S_4), multiplets of weight 4 (weight $k \leq 10$) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N = 3$ (i.e., A_4) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N = 5$ (i.e., A_5), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

The modular forms of level $N = 2, 3, 4$ for $\Gamma_{2,3,4} \simeq S_3, A_4, S_4$ have been constructed by use of the Dedekind eta function, $\eta(\tau)$,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{i2\pi\tau}.$$

For A_4, $\eta(3\tau)$, $\eta(\tau/3)$, $\eta((\tau+1)/3)$ and $\eta((\tau+2)/3)$ were used.

F. Feruglio, arXiv:1706.08749

For S_4, $\eta(\tau + 1/2)$, $\eta(4\tau)$, $\eta(\tau/4)$, $\eta((\tau+1)/4)$, $\eta((\tau+2)/4)$ and $\eta((\tau+3)/4)$ were used.

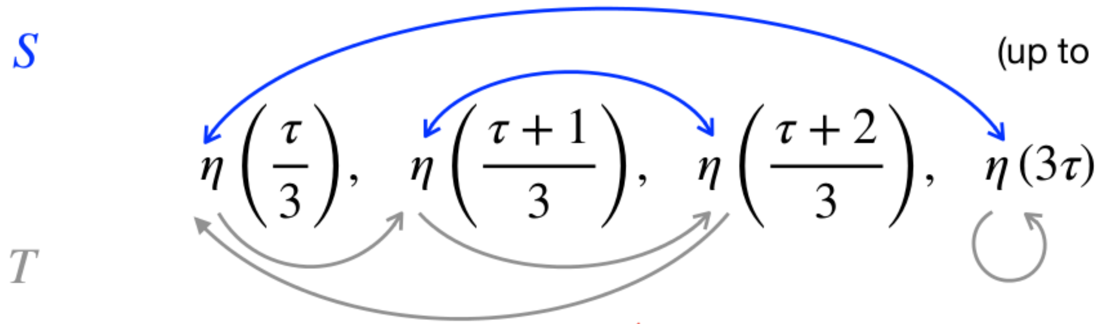
J.T. Penedo, STP, arXiv:1806.11040

Modular forms of weight 2

Level *N* = 3 $(\Gamma_3 \simeq A_4 : \quad S^2 = (ST)^3 = T^3 = I)$

N\k	0	2	4	6
3	1	**3**	5	7

(up to multiplicative factors)



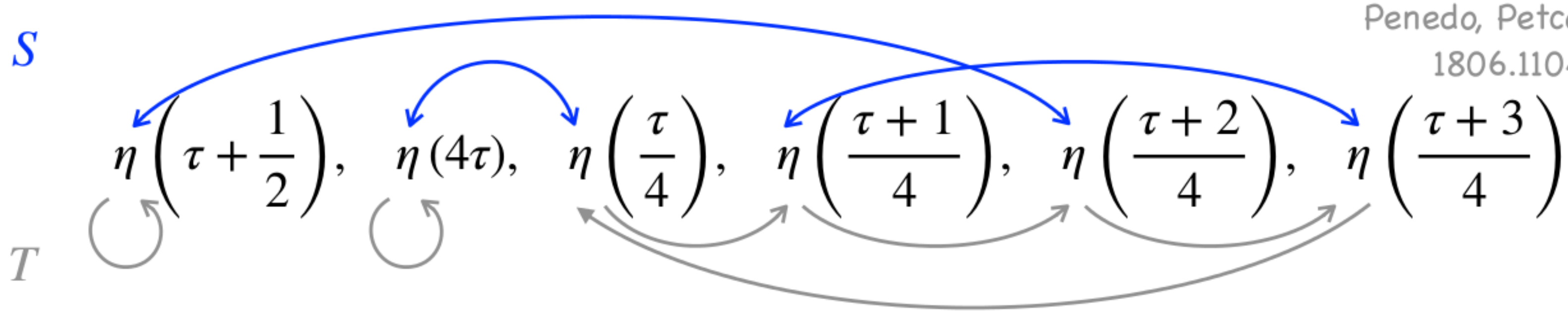
Feruglio, 1706.08749

A_4 triplet of weight 2 modular forms

Level *N* = 4 $(\Gamma_4 \simeq S_4 : \quad S^2 = (ST)^3 = T^4 = I)$

N\k	0	2	4	6
4	1	**5**	9	13

Penedo, Petcov, 1806.11040



S_4 doublet and triplet ($\mathbf{3}'$) of weight 2 modular forms

From A. Titov, talk at FLASY 2019

The modular forms of the weight 2 are expressed in terms of $\eta'(\tau+1/2)/\eta(\tau+1/2)$, $\eta'(4\tau)/\eta(4\tau)$, $\eta'(\tau/4)/\eta(\tau/4)$, $\eta'((\tau+1)/4)/\eta((\tau+1)/4)$, $\eta'((\tau+2)/4)/\eta((\tau+2)/4)$, $\eta'((\tau+3)/4)/\eta((\tau+3)/4)$, **and are written as:**

$$Y_1(\tau) = Y(1, 1, \omega, \omega^2, \omega, \omega^2 | \tau),$$

$$Y_2(\tau) = Y(1, 1, \omega^2, \omega, \omega^2, \omega | \tau),$$

$$Y_3(\tau) = Y(1, -1, -1, -1, 1, 1 | \tau),$$

$$Y_4(\tau) = Y(1, -1, -\omega^2, -\omega, \omega^2, \omega | \tau),$$

$$Y_5(\tau) = Y(1, -1, -\omega, -\omega^2, \omega, \omega^2 | \tau)\,, \quad \omega = e^{i2\pi/3}\,.$$

J.T. Penedo, STP, arXiv:1806.11040

$Y(a_1, a_2, a_3, a_4, a_5, a_6|\tau)$ **is given by:**

$$Y(a_1, a_2, a_3, a_4, a_5, a_6|\tau) \;=\; a_1 \frac{\eta'(\tau + 1/2)}{\eta(\tau + 1/2)} + 4a_2 \frac{\eta'(4\tau)}{\eta(4\tau)} + \frac{1}{4}\sum_{m=0}^{3} a_{m+3} \frac{\eta'((\tau + m)/4)}{\eta((\tau + m)/4)}$$

where $\sum_{i=1}^{6} a_i = 0$**. The five independent modular form combinations in preceding equations are decomposed into the** $\mathbf{2}$ **and** $\mathbf{3'}$ **irreducible representations of** S_4**:**

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5((\tau) \end{pmatrix}.$$

J.T. Penedo, STP, arXiv:1806.11040

The modular forms of higher weight transform according to certain irreps of S_4. The dimension of the linear space of mod. forms of weight k is dim$\mathcal{M}_k(\Gamma(4)) = 2k+1$. **At weight 4 there are 9 independent modular forms transforming in the $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{3}'$ irreps of S_4:**

$$Y_{\mathbf{1}}^{(4)} = Y_1 Y_2\,, \qquad Y_{\mathbf{2}}^{(4)} = \begin{pmatrix} Y_2^2 \\ Y_1^2 \end{pmatrix},$$

$$Y_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1 Y_4 - Y_2 Y_5 \\ Y_1 Y_5 - Y_2 Y_3 \\ Y_1 Y_3 - Y_2 Y_4 \end{pmatrix}, \qquad Y_{\mathbf{3}'}^{(4)} = \begin{pmatrix} Y_1 Y_4 + Y_2 Y_5 \\ Y_1 Y_5 + Y_2 Y_3 \\ Y_1 Y_3 + Y_2 Y_4 \end{pmatrix}.$$

J.T. Penedo, STP, arXiv:1806.11040

For the case of $N = 4$ (i.e., S_4) we are going to consider further the weight 2 and the higher weight $k \leq 10$ modular multiplets have been computed in the basis of S and T generators employed in arXiv:1806.11040. In this basis the triplet irreps of S and T to be used in our analysis read:

$$S = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}, \; T = \pm\frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix},$$

$\omega = e^{i2\pi\tau/3}$. The plus (minus) corresponds to the irrep $\mathbf{3}$ ($\mathbf{3'}$) of S_4.

In the employed basis we have:

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

In certain cases of $N = 3, 4, 5$ (i.e., A_4, S_4, A_5) it proves convenient to work in basis in which the generators S and T of these groups are represented by symmetric matrices,

$$\rho_{\mathbf{r}}(S) = \rho_{\mathbf{r}}^T(S)\,, \qquad \rho_{\mathbf{r}}(T) = \rho_{\mathbf{r}}^T(T)\,,$$

for all irreducible representations $\mathbf{r}$.
The modular forms of levels $N = 3, 4, 5$ and weights $k \leq 10$ in the symmetric bases for S and T can be found in P.P. Novichkov et al., arXiv:1905.11970.
We will be interested in the finite modular group $\Gamma_4 \simeq S_4$.

Residual Symmetries

Residual symmetries arise whenever the VEV of the modulus τ breaks $\overline{\Gamma}$ only partially, i.e., the little group (stabiliser) of $\langle\tau\rangle$ is non-trivial. There are only 3 inequivalent finite points with non-trivial little groups of $\overline{\Gamma}$:
$\langle\tau\rangle = \mp 1/2 + i\sqrt{3}/2 \equiv \tau_{L,R}$, **and** $\langle\tau\rangle = i \equiv \tau_C$.

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Left cusp of the fundamental domain,
$\langle\tau\rangle = -1/2 + i\sqrt{3}/2 \equiv \tau_L$, **is invariant under**
ST: $\tau = -1/(\tau+1)$, $Z_3^{ST} = \{I, ST, (ST)^2\}$ **symmetry.**

The $\langle\tau\rangle = i$ point is invariant under
S **transformation** $\tau = -1/\tau$: $Z_2^S = \{I, S\}$ **symmetry.**

Right cusp of the fundamental domain,
$\langle\tau\rangle = 1/2 + i\sqrt{3}/2 \equiv \tau_R = T\,\tau_L$ **is invariant under**
TS: $\tau = 1 - 1/\tau$, $Z_3^{TS} = \{I, TS, (TS)^2\}$ **symmetry.**

There is also infinite point $\langle\tau\rangle = i\infty \equiv \tau_T$, in which the subgroup $Z_3^T = \{I, T, T^2\}$ **of** A_4, $Z_4^T = \{I, T, T^2, T^3\}$ **of** S_4 ($Z_N^T = \{I, T, T^2, T^3 ..., T^{N-1}\}$ **of** Γ_N) **is preserved.**

The Framework

$\mathcal{N}=1$ **rigid SUSY, the matter action $\mathcal{S}$ reads:**

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\;\; K(\tau,\overline{\tau},\psi,\overline{\psi}) + \left(\int \mathrm{d}^4x\, \mathrm{d}^2\theta\;\; W(\tau,\psi) + \mathrm{h.c.}\right)\,,$$

K **is the Kähler potential,** W **is the superpotential,** ψ **denotes a set of chiral supermultiplets** ψ_i**,** θ **and** $\overline{\theta}$ **are Grassmann variables;**
τ **is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation** τ **for the lowest complex scalar component of the modulus superfield and call this component also "modulus").**
τ **and** ψ_i **transform under the action of** $\overline{\Gamma}$ **in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that** $\psi_i=\psi_i(x)$ **transform in a certain irrep** r_i **of** Γ_N**, the transformations read:**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}\, : \qquad \begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d}\,, \\ \psi_i \to (c\tau+d)^{-k_i}\,\rho_{\mathbf{r}_i}(\gamma)\,\psi_i\,. \end{cases}$$

ψ_i **is not a multiplet of modular forms,** $(-k_i)$ **can be odd and/or negative.**
Invariance of $\mathcal{S}$ **under these transformations implies (global SUSY):**

$$\begin{cases} W(\tau, \psi) \to W(\tau, \psi)\,, \\ K(\tau, \overline{\tau}, \psi, \overline{\psi}) \to K(\tau, \overline{\tau}, \psi, \overline{\psi}) + f_K(\tau, \psi) + \overline{f_K}(\overline{\tau}, \overline{\psi})\,. \end{cases}$$

The second line represents a Kähler transformation.
An example Kähler potential we will use in what follows, reads:

$$K(\tau, \overline{\tau}, \psi, \overline{\psi}) = -\Lambda_0^2 \log(-i\tau + i\overline{\tau}) + \sum_i \frac{|\psi_i|^2}{(-i\tau + i\overline{\tau})^{k_i}}\,,$$

Λ_0 having mass dimension one. Under Γ_N, $(\tau - \overline{\tau}) \to (\tau - \overline{\tau})|c\tau + d|^{-2}$, and

$$K(\tau, \overline{\tau}, \psi, \overline{\psi}) \to K(\tau, \overline{\tau}, \psi, \overline{\psi}) + \Lambda_0^2\,(\log(c\tau + d) + \log(c\overline{\tau} + d))$$

Assuming only τ acquires a VEV, K gives rise to kinetic terms for the scalar components of supermultiplets τ and ψ of the form:

$$\frac{\Lambda_0^2}{(-i\tau + i\overline{\tau})^2} \partial_\mu \overline{\tau} \partial^\mu \tau + \sum_i \frac{\partial_\mu \overline{\psi_i} \partial^\mu \psi_i}{(-i\tau + i\overline{\tau})^{k_i}}\,.$$

The superpotential can be expanded in powers of ψ_i:

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_s g_{i_1\dots i_n,s} \left(Y_{i_1\dots i_n,s}(\tau)\,\psi_{i_1}\dots\psi_{i_n}\right)_{\mathbf{1},s} ,$$

$\mathbf{1}$ stands for an invariant singlet of Γ_N. For each set of n fields $\{\psi_{i_1},\dots,\psi_{i_n}\}$, the index s labels the independent singlets. Each of these is accompanied by a coupling constant $g_{i_1\dots i_n,s}$ and is obtained using a modular multiplet $Y_{i_1\dots i_n,s}$ of the requisite weight. To ensure invariance of W under Γ_N, $Y_{i_1\dots i_n,s}(\tau)$ must transform as:

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)\,Y(\tau)\,,$$

$\mathbf{r}_Y$ is a representation of Γ_N, and k_Y and $\mathbf{r}_Y$ are such that

$$k_Y = k_{i_1} + \cdots + k_{i_n}\,, \tag{1}$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \dots \otimes \mathbf{r}_{i_n} \supset \mathbf{1}\,. \tag{2}$$

Thus, $Y_{i_1\dots i_n,s}(\tau)$ represents a multiplet of weight k_Y and level N modular forms transforming in the representation $\mathbf{r}_Y$ of Γ_N.

Lepton Flavour Models Based on S_4 (Seesaw Models without Flavons)

We assume that neutrino masses originate from the (supersymmetric) type I seesaw mechanism. The superpotential in the lepton sector reads

$$W = \alpha\,(E^c L H_d f_E\,(Y))_1 + g\,(N^c L H_u f_N\,(Y))_1 + \Lambda\,(N^c N^c f_M\,(Y))_1\ ,$$

a sum over all independent invariant singlets with the coefficients $\alpha = (\alpha, \alpha', \ldots)$, $g = (g, g', \ldots)$ and $\Lambda = (\Lambda, \Lambda', \ldots)$ is implied. $f_{E,N,M}(Y)$ denote the modular form multiplets required to ensure modular invariance.

For simplicity, we make the following assumptions:

- **Higgs doublets H_u and H_d transform trivially under Γ_4, $\rho_u = \rho_d \sim \mathbf{1}$, and $k_u = k_d = 0$;**
- **lepton $SU(2)$ doublets L_1, L_2, L_3 furnish a 3-dim. irrep of Γ_4, i.e., $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$;**
- **neutral lepton gauge singlets N_1^c, N_2^c, N_3^c transform as a triplet of Γ_4, $\rho_N \sim \mathbf{3}$ or $\mathbf{3}'$;**
- **charged lepton $SU(2)$ singlets E_1^c, E_2^c, E_3^c transform as singlets of Γ_4, $\rho_{1,2,3} \sim \mathbf{1},\ \mathbf{1}'$.**

With these assumptions, we can rewrite the superpotential as

$$W = \sum_{i=1}^{3} \alpha_i\,(E_i^c\,L\,f_{E_i}\,(Y))_1\,H_d + g\,(N^c\,L\,f_N\,(Y))_1\,H_u + \Lambda\,(N^c\,N^c\,f_M\,(Y))_1\ ,$$

Assigning weights $(-k_i)$, $(-k_L)$, $(-k_N)$ to E_i^c, L, N^c, and weights k_{α_i}, k_g, k_Λ to the multiplets of modular forms $f_{E_i}(Y)$, $f_N(Y)$, $f_M(Y)$, modular invariance of the superpotential requires

$$\begin{cases} k_{\alpha_i} = k_i + k_L \\ k_g = k_N + k_L \\ k_\Lambda = 2\,k_N \end{cases} \quad \Leftrightarrow \quad \begin{cases} k_i = k_{\alpha_i} - k_g + k_\Lambda/2 \\ k_L = k_g - k_\Lambda/2 \\ k_N = k_\Lambda/2 \end{cases} .$$

By specifying the weights of the modular forms one obtains the weights of the matter superfields.

After modular symmetry breaking, the matrices of charged lepton and neutrino Yukawa couplings, λ and $\mathcal{Y}$, as well as the Majorana mass matrix M for heavy neutrinos, are generated:

$$W = \lambda_{ij}\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}\, N_i^c\, L_j\, H_u + \frac{1}{2}\, M_{ij}\, N_i^c\, N_j^c\,,$$

a sum over $i,j = 1,2,3$ is assumed. After integrating out N^c and after EWS breaking, the charged lepton mass matrix M_e and the light neutrino Majorana mass matrix M_ν are generated (we work in the L-R convention for the charged lepton mass term and the R-L convention for the light and heavy neutrino Majorana mass terms):

$$M_e = v_d\, \lambda^\dagger\,, \qquad v_d \equiv H_d^0\,,$$
$$M_\nu = -v_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}\,, \qquad v_u \equiv H_u^0\,.$$

The Majorana mass term for heavy neutrinos

Assume $k_\Lambda = 0$, i.e., no non-trivial modular forms are present in $\Lambda\,(N^c\,N^c\,f_M\,(Y))_{\mathbf{1}}$, $k_N = 0$, and for both $\rho_N \sim \mathbf{3}$ or $\rho_N \sim \mathbf{3'}$

$$(N^c\,N^c)_{\mathbf{1}} = N_1^c\,N_1^c + N_2^c\,N_3^c + N_3^c\,N_2^c\,,$$

leading to the following mass matrix for heavy neutrinos:

$$M = 2\,\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \textbf{for} \quad k_\Lambda = 0\,.$$

The spectrum of heavy neutrino masses is degenerate; the only free parameter is the overall scale Λ, which can be rendered real. The Majorana mass term conserves a "non-standard" lepton charge and two of the three heavy Majorana neutrinos with definite mass form a Dirac pair.

C.N. Leung, STP, 1983

The neutrino Yukawa couplings

The lowest non-trivial weight, $k_g = 2$, leads to

$$g\,(N^c\,L\,Y_{\mathbf{2}})_{\mathbf{1}}\,H_u + g'\,(N^c\,L\,Y_{\mathbf{3'}})_{\mathbf{1}}\,H_u\,.$$

There are 4 possible assignments of ρ_N and ρ_L we consider. Two of them, namely $\rho_N = \rho_L \sim \mathbf{3}$ and $\rho_N = \rho_L \sim \mathbf{3'}$ give the following form of $\mathcal{Y}$:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 0 & Y_5 & -Y_4 \\ -Y_5 & 0 & Y_3 \\ Y_4 & -Y_3 & 0 \end{pmatrix}\right], \quad \text{for} \quad k_g = 2 \quad \text{and} \quad \rho_N = \rho_L\,.$$

The two remaining combinations, $(\rho_N, \rho_L) \sim (\mathbf{3}, \mathbf{3'})$ and $(\mathbf{3'}, \mathbf{3})$, lead to:

$$\mathcal{Y} = g\left[\begin{pmatrix} 0 & -Y_1 & Y_2 \\ -Y_1 & Y_2 & 0 \\ Y_2 & 0 & -Y_1 \end{pmatrix} + \frac{g'}{g}\begin{pmatrix} 2Y_3 & -Y_5 & -Y_4 \\ -Y_5 & 2Y_4 & -Y_3 \\ -Y_4 & -Y_3 & 2Y_5 \end{pmatrix}\right], \quad \text{for} \quad k_g = 2 \quad \text{and} \quad \rho_N \neq \rho_L\,.$$

In both cases, up to an overall factor, the matrix $\mathcal{Y}$ depends on one complex parameter g'/g and the VEV τ.

The charged lepton Yukawa couplings

Since we consider $\rho_i \sim \mathbf{1}$ or $\mathbf{1}'$ and $\rho_L \sim \mathbf{3}$ or $\mathbf{3}'$, we have four possible combinations $\rho_i \otimes \rho_L$. None of them contain the invariant singlet. Thus, the weights k_{α_i} cannot be zero, i.e., they are strictly positive, $k_{\alpha_i} > 0$. Moreover, $f_{E_i}(Y)$ should transform in $\mathbf{3}$ if $(\rho_i, \rho_L) \sim (\mathbf{1}, \mathbf{3})$ or $(\mathbf{1}', \mathbf{3}')$, and in $\mathbf{3}'$ if $(\rho_i, \rho_L) \sim (\mathbf{1}, \mathbf{3}')$ or $(\mathbf{1}', \mathbf{3})$. Thus, for each $i = 1, 2, 3$, we have

$$\alpha_i \left(E_i^c\, L\, f_{E_i}(Y)\right)_{\mathbf{1}} H_d = E_i^c \sum_a \alpha_{i,a} \left[L_1 \left(Y_a^{(k_{\alpha_i})}\right)_1 + L_2 \left(Y_a^{(k_{\alpha_i})}\right)_3 + L_3 \left(Y_a^{(k_{\alpha_i})}\right)_2\right] H_d \,,$$

where $Y_a^{(k_{\alpha_i})}$ are independent triplets ($\mathbf{3}$ or $\mathbf{3}'$ depending on ρ_i and ρ_L) of weight k_{α_i}. $k_{\alpha_i} = 2$, $i = 1, 2, 3$ or $i = 1, 2$ is not phenomenologically viable (leads to two or one zero mass charged leptons). The minimal (in terms of weights) viable possibility is defined by $k_{\alpha_i} = 2$ and $k_{\alpha_j} = k_{\alpha_p} = 4$, for $j \neq p$, with $\rho_j \neq \rho_p$. Possible since there are two triplets of weight 4, $Y_{\mathbf{3}}^{(4)}$ and $Y_{\mathbf{3}'}^{(4)}$.

Then the relevant part of W, W_e, can take 6 different forms which lead to the same matrix U_e diagonalising $M_e M_e^\dagger = v_d^2\, \lambda^\dagger \lambda$, and thus do not lead to new results for the PMNS matrix. We give just one of these 6 forms corresponding to $\rho_L = \mathbf{3}$, $\rho_1 = \mathbf{1}'$, $\rho_2 = \mathbf{1}$, $\rho_3 = \mathbf{1}'$:

$$\alpha \left(E_1^c\, L\, Y_{\mathbf{3}'}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c\, L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c\, L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d \,.$$

This leads leads to

$$\lambda = \begin{pmatrix} \alpha\, Y_3 & \alpha\, Y_5 & \alpha\, Y_4 \\ \beta\,(Y_1 Y_4 - Y_2 Y_5) & \beta\,(Y_1 Y_3 - Y_2 Y_4) & \beta\,(Y_1 Y_5 - Y_2 Y_3) \\ \gamma\,(Y_1 Y_4 + Y_2 Y_5) & \gamma\,(Y_1 Y_3 + Y_2 Y_4) & \gamma\,(Y_1 Y_5 + Y_2 Y_3) \end{pmatrix},$$

In this "minimal" example the matrix λ depends on 3 free parameters, α, β and γ, which can be rendered real by re-phasing of the charged lepton fields, and the complex τ.

Numerical Analysis

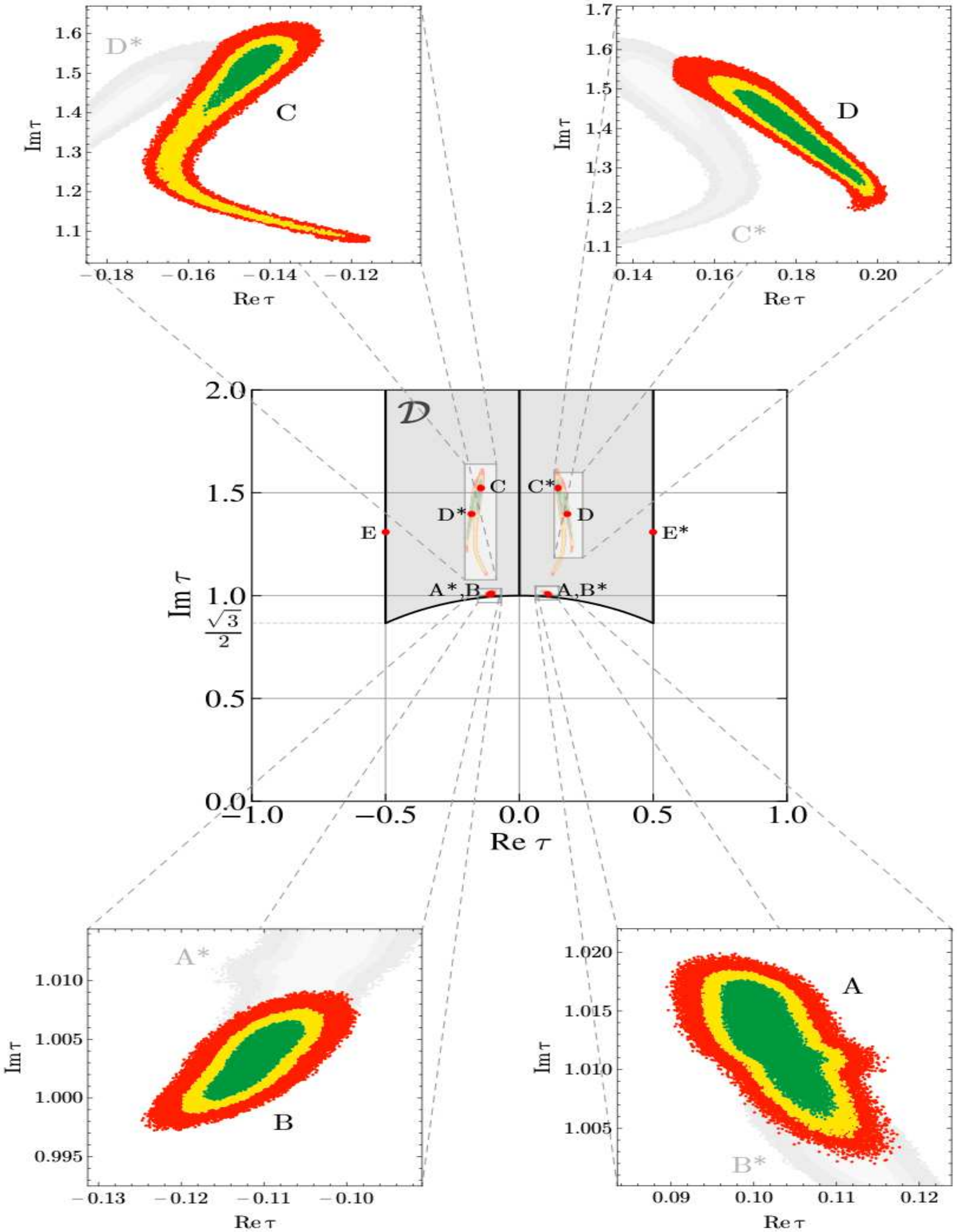
Each model depends on a set of dimensionless parameters

$$p_i = (\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots)\,,$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: $v_d\,\alpha$ for M_e and $v_u^2\,g^2/\Lambda$ for M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with the experimental results summarised in the Table below. We assume also to be in a regime in which the running of neutrino parameters is negligible.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \text{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\lvert\Delta m^2\rvert/(10^{-3}\ \text{eV}^2)$	$2.455^{+0.035}_{-0.032}$	$2.441^{+0.033}_{-0.035}$
$r \equiv \delta m^2/\lvert\Delta m^2\rvert$	0.0299 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	$0.304^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0214^{+0.0009}_{-0.0007}$	$0.0218^{+0.0008}_{-0.0007}$
$\sin^2\theta_{23}$	$0.551^{+0.019}_{-0.070}$	$0.557^{+0.017}_{-0.024}$
δ/π	$1.32^{+0.23}_{-0.18}$	$1.52^{+0.14}_{-0.15}$

Best fit values and 1σ ranges for neutrino oscillation parameters, obtained in the global analysis of F. Capozzi et al., arXiv:1804.09678, and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in F. Feruglio, arXiv:1706.08749 obtained from G.G. Ross and M. Serna, arXiv:0704.1248. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$. The best fit value and 1σ range of δ did not drive the numerical searches here reported.



P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933

	Best fit value	2σ range	3σ range
Re τ	± 0.1045	$\pm(0.09597 - 0.1101)$	$\pm(0.09378 - 0.1128)$
Im τ	1.01	$1.006 - 1.018$	$1.004 - 1.018$
β/α	9.465	$8.247 - 11.14$	$7.693 - 12.39$
γ/α	0.002205	$0.002032 - 0.002382$	$0.001941 - 0.002472$
Re g'/g	0.233	$-0.02383 - 0.387$	$-0.02544 - 0.4417$
Im g'/g	± 0.4924	$\pm(-0.592 - 0.5587)$	$\pm(-0.6046 - 0.5751)$
$v_d\,\alpha$ [MeV]	53.19		
$v_u^2\,g^2/\Lambda$ [eV]	0.00933		
m_e/m_μ	0.004802	$0.004418 - 0.005178$	$0.00422 - 0.005383$
m_μ/m_τ	0.0565	$0.048 - 0.06494$	$0.04317 - 0.06961$
r	0.02989	$0.02836 - 0.03148$	$0.02759 - 0.03224$
δm^2 [10^{-5} eV^2]	7.339	$7.074 - 7.596$	$6.935 - 7.712$
$\|\Delta m^2\|$ [10^{-3} eV^2]	2.455	$2.413 - 2.494$	$2.392 - 2.513$
$\sin^2\theta_{12}$	0.305	$0.2795 - 0.3313$	$0.2656 - 0.3449$
$\sin^2\theta_{13}$	0.02125	$0.01988 - 0.02298$	$0.01912 - 0.02383$
$\sin^2\theta_{23}$	0.551	$0.4846 - 0.5846$	$0.4838 - 0.5999$
Ordering	NO		
m_1 [eV]	0.01746	$0.01196 - 0.02045$	$0.01185 - 0.02143$
m_2 [eV]	0.01945	$0.01477 - 0.02216$	$0.01473 - 0.02307$
m_3 [eV]	0.05288	$0.05099 - 0.05405$	$0.05075 - 0.05452$
$\sum_i m_i$ [eV]	0.0898	$0.07774 - 0.09661$	$0.07735 - 0.09887$
$\|\langle m\rangle\|$ [eV]	0.01699	$0.01188 - 0.01917$	$0.01177 - 0.02002$
δ/π	± 1.314	$\pm(1.266 - 1.95)$	$\pm(1.249 - 1.961)$
α_{21}/π	± 0.302	$\pm(0.2821 - 0.3612)$	$\pm(0.2748 - 0.3708)$
α_{31}/π	± 0.8716	$\pm(0.8162 - 1.617)$	$\pm(0.7973 - 1.635)$
$N\sigma$	0.02005		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases A and A*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.1045 + i\,1.01$).

	Best fit value	2σ range	3σ range
$\text{Re}\,\tau$	∓ 0.1435	$\mp(0.137 - 0.1615)$	$\mp(0.1222 - 0.168)$
$\text{Im}\,\tau$	1.523	1.147 – 1.572	1.088 – 1.594
β/α	17.82	10.99 – 21.38	9.32 – 23.66
γ/α	0.003243	0.002518 – 0.003565	0.00227 – 0.003733
$\text{Re}\,g'/g$	−0.8714	−(0.8209 – 1.132)	−(0.7956 – 1.148)
$\text{Im}\,g'/g$	∓ 2.094	$\mp(1.439 - 2.157)$	$\mp(1.409 - 2.182)$
$v_d\,\alpha$ [MeV]	71.26		
$v_u^2\,g^2/\Lambda$ [eV]	0.008173		
m_e/m_μ	0.004797	0.00442 – 0.005183	0.004215 – 0.005378
m_μ/m_τ	0.05655	0.04806 – 0.06507	0.04348 – 0.0698
r	0.0301	0.02857 – 0.03162	0.0278 – 0.03246
δm^2 [10^{-5} eV^2]	7.346	7.084 – 7.589	6.946 – 7.717
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.44	2.4 – 2.479	2.377 – 2.498
$\sin^2\theta_{12}$	0.303	0.278 – 0.3288	0.2657 – 0.3436
$\sin^2\theta_{13}$	0.02175	0.02035 – 0.0234	0.01957 – 0.0242
$\sin^2\theta_{23}$	0.5571	0.4905 – 0.588	0.4551 – 0.6026
Ordering	IO		
m_1 [eV]	0.0513	0.04938 – 0.0518	0.04882 – 0.05207
m_2 [eV]	0.05201	0.05012 – 0.05248	0.04958 – 0.05274
m_3 [eV]	0.01512	0.00576 – 0.01594	0.00316 – 0.0163
$\sum_i m_i$ [eV]	0.1184	0.1053 – 0.1201	0.102 – 0.1208
$\lvert\langle m\rangle\rvert$ [eV]	0.0263	0.0239 – 0.04266	0.02288 – 0.04551
δ/π	± 1.098	$\pm(1.026 - 1.278)$	$\pm(0.98 - 1.289)$
α_{21}/π	± 1.241	$\pm(1.162 - 1.651)$	$\pm(1.113 - 1.758)$
α_{31}/π	± 0.2487	$\pm(0.1474 - 0.3168)$	$\pm(0.069 - 0.346)$
$N\sigma$	0.0357		

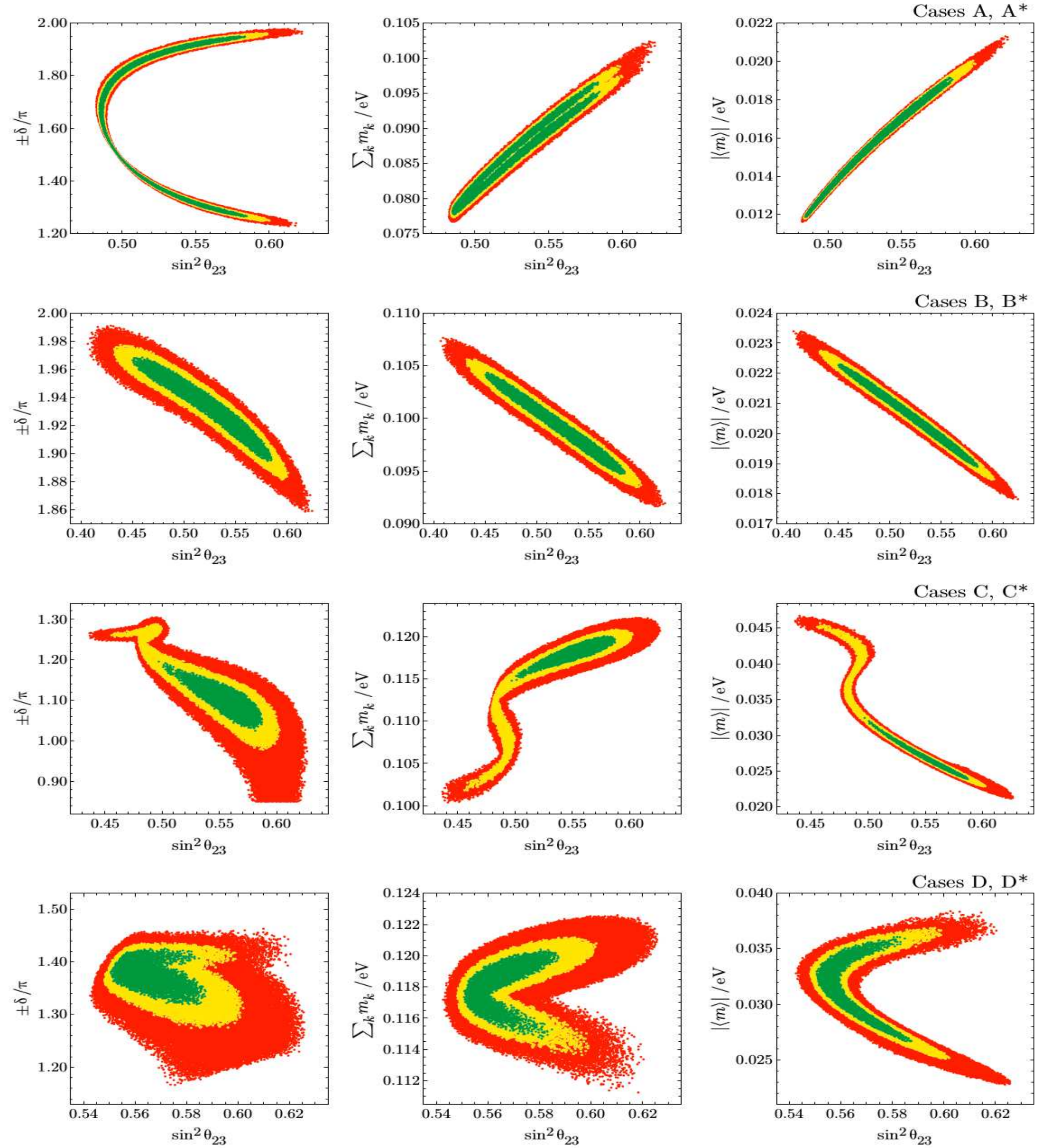
Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases C and C*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.1453 + i\,1.523$).

	Best fit value	2σ range	3σ range
$\text{Re}\,\tau$	± 0.179	$\pm(0.165 - 0.1963)$	$\pm(0.1589 - 0.199)$
$\text{Im}\,\tau$	1.397	1.262 – 1.496	1.236 – 1.529
β/α	15.35	11.67 – 18.66	10.79 – 21.09
γ/α	0.002924	0.002582 – 0.003289	0.002443 – 0.003459
$\text{Re}\,g'/g$	-1.32	$-(1.189 - 1.438)$	$-(1.131 - 1.447)$
$\text{Im}\,g'/g$	± 1.733	$\pm(1.357 - 1.948)$	$\pm(1.306 - 2.017)$
$v_d\,\alpha$ [MeV]	68.42		
$v_u^2\,g^2/\Lambda$ [eV]	0.00893		
m_e/m_μ	0.004786	0.004431 – 0.005186	0.004221 – 0.005386
m_μ/m_τ	0.0554	0.0481 – 0.06502	0.04343 – 0.06968
r	0.03023	0.02859 – 0.03163	0.02775 – 0.03244
δm^2 [10^{-5} eV^2]	7.367	7.088 – 7.59	6.937 – 7.713
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.437	2.4 – 2.479	2.378 – 2.499
$\sin^2\theta_{12}$	0.3031	0.2791 – 0.3286	0.2657 – 0.3436
$\sin^2\theta_{13}$	0.02184	0.02038 – 0.02337	0.01954 – 0.0242
$\sin^2\theta_{23}$	0.5577	0.5509 – 0.5869	0.5482 – 0.6013
Ordering	IO		
m_1 [eV]	0.05122	0.05051 – 0.05185	0.05023 – 0.05212
m_2 [eV]	0.05193	0.05125 – 0.05253	0.05098 – 0.05279
m_3 [eV]	0.01495	0.01293 – 0.01613	0.01223 – 0.01649
$\sum_i m_i$ [eV]	0.1181	0.1149 – 0.1203	0.1139 – 0.1212
$\lvert\langle m\rangle\rvert$ [eV]	0.03104	0.02666 – 0.03597	0.02515 – 0.03677
δ/π	± 1.384	$\pm(1.32 - 1.4245)$	$\pm(1.271 - 1.437)$
α_{21}/π	± 1.343	$\pm(1.227 - 1.457)$	$\pm(1.171 - 1.479)$
α_{31}/π	± 0.806	$\pm(0.561 - 1.092)$	$\pm(0.448 - 1.149)$
$N\sigma$	0.3811		

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases D and D*, (which refer to $(k_\Lambda, k_g) = (0,2)$ and $\tau = \pm 0.179 + i\,1.397$).

	Best fit value	3σ range
Re τ	∓ 0.4996	$\mp(0.48 - 0.5084)$
Im τ	1.309	$1.246 - 1.385$
β/α	0.000243	$0.0002004 - 0.0002864$
γ/α	0.03335	$0.02799 - 0.03926$
Re g'/g	-0.06454	$-(0.01697 - 0.1215)$
Im g'/g	∓ 0.569	$\mp(0.4572 - 0.6564)$
$v_d\,\alpha$ [MeV]	1125	
$v_u^2\,g^2/\Lambda$ [eV]	0.0174	
m_e/m_μ	0.004797	$0.004393 - 0.005197$
m_μ/m_τ	0.05626	$0.04741 - 0.0654$
r	0.02985	$0.02826 - 0.03146$
δm^2 [10^{-5} eV^2]	7.332	$7.055 - 7.593$
$\lvert\Delta m^2\rvert$ [10^{-3} eV^2]	2.456	$2.413 - 2.497$
$\sin^2\theta_{12}$	0.311	$0.2895 - 0.3375$
$\sin^2\theta_{13}$	0.02185	$0.02041 - 0.02351$
$\sin^2\theta_{23}$	0.4469	$0.43 - 0.4614$
Ordering	NO	
m_1 [eV]	0.01774	$0.01703 - 0.01837$
m_2 [eV]	0.0197	$0.01906 - 0.02025$
m_3 [eV]	0.05299	$0.05251 - 0.05346$
$\sum_i m_i$ [eV]	0.09043	$0.08874 - 0.09195$
$\lvert\langle m\rangle\rvert$ [eV]	0.006967	$0.006482 - 0.007288$
δ/π	± 1.601	$\pm(1.287 - 1.828)$
α_{21}/π	± 1.093	$\pm(0.8593 - 1.178)$
α_{31}/π	± 0.7363	$\pm(0.3334 - 0.9643)$
$N\sigma$	2.147	

Best fit values along with 2σ and 3σ ranges of the parameters and observables in cases E and E*, (which refer to $(k_\Lambda, k_g) = (0, 2)$ and $\tau = \pm 0.4996 + i\,1.309$).



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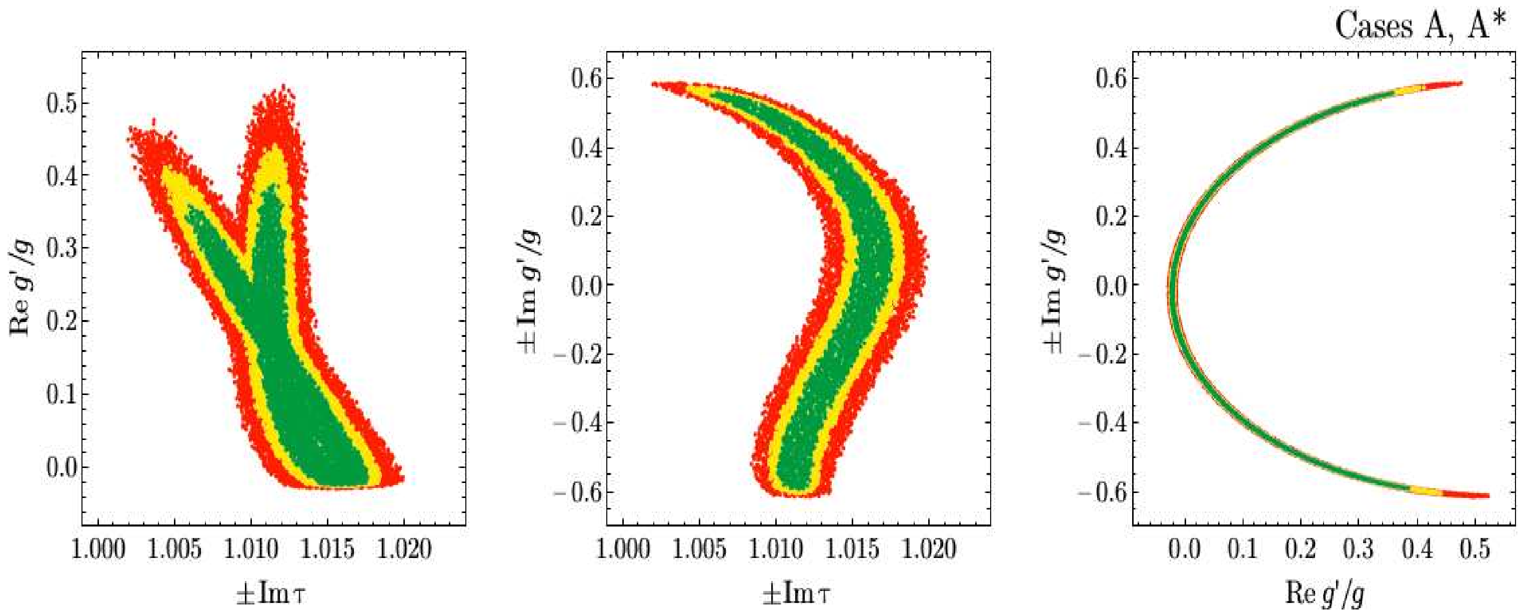

P.P. Novichkov et al., arXiv:1811.04933

Conclusions.

• Understanding the origin of the pattern of neutrino mixing and of neutrino mass squared differences that emerged from the neutrino oscillation data in the recent years is one of the most challenging problems in neutrino physics. It is part of the fundamental problem of understanding the origin of flavour in particle physics.

• The modular invariance (finite modular group symmetries) is a new elegant and very promising approach to the flavour problem. It has been successfully applied to the lepton flavour problem.

• In its minimal version the approach involves just one complex scalar field – the modulus τ, and a certain rather small number of constant parameters. The modular symmetry is broken by the the VEV of τ.

• The models of lepton flavour based of finite modular symmetries, lead to testable predictions for $\min(m_j)$, type of the neutrino mass spectrum (NO or IO), $\sum_i m_i$, the CPV Dirac and Majorana phases, $|\langle m \rangle|$, θ_{23}, as well as of correlations between different observables.

• The modular invariance approach to the flavour problem is still at the early stage of its development, with many aspects still to be understood.