Walter Burke Institute for
Theoretical Physics

On the Singularity Structure of Maximally Supersymmetric Scattering Amplitudes

Jaroslav Trnka
Caltech

N. Arkani-Hamed, J. Bourjaily, F. Cachazo, JT, 1410.0354

Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, JT, to appear

Motivation

- Goal: New mathematical structures in QFT.
- Ideal test objects: **on-shell scattering amplitudes**.
- Standard approach
  1. Integrand of the amplitude.
  2. Integrated expression.
  3. Cross sections,...
- New formulation of QFT $\rightarrow$ most visible in the **integrand**.
- $\mathcal{N} = 4$ SYM : harmonic oscillator of 21^{th} century.

Overview

- Amplitudes in planar $\mathcal{N} = 4$ SYM
  - New structures and symmetries, dual formulation.
  - **Logarithmic singularities**.

- Amplitudes in full $\mathcal{N} = 4$ SYM
  - No symmetries, no dual formulation,...
  - Evidence for 4pt: logarithmic singularities.

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Conjecture: [Arkani-Hamed, Bourjaily, Cachazo, JT]
Four point amplitudes in $\mathcal{N}=4$ SYM have only logarithmic singularities and no poles at infinity.

- In the planar sector: it implies dual conformal invariance.

[Bern, Herrmann, Litsey, Stankowicz, JT]

Planar $\mathcal{N} = 4$ SYM

- Unitary methods successful: high loop results, BDS,...

[Bern, Dixon, Kosower, Roiban, Carrasco, Johansson, Smirnov,...]

- Recursion relations

[Britto, Cachazo, Feng, Witten; Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT]

- Amplitudes at strong coupling

[Alday, Maldacena]

- Dual conformal symmetry

[Drummond, Henn, Korchemsky, Smirnov, Sokatchev; Alday, Maldacena]

- Yangian symmetry

[Drummond, Henn, Plefka]

Planar $\mathcal{N} = 4$ SYM

- Amplitudes/super-Wilson loops correspondence
  [Caron-Huot; Mason, Skinner]

- Gamma cusp to all orders
  [Beisert, Eden, Staudacher]

- OPE and flux-tube S-matrix
  [Basso, Sever, Vieira]

- Integrals, symbols, polylogs
  [Henn, Smirnov^2,...; Golden, Goncharov, Paulos, Spradlin, Volovich]

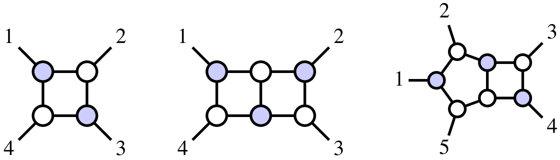
- Hexagon bootstrap
  [Dixon, Drummond, Duhr, Henn, von Hippel, Pennington]

- Many more. . . .

Dual formulation

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT]

- On-shell physics
  - Feynman diagrams: off-shell physics.
  - Unitary methods: on-shell data.
  - **On-shell diagrams**: on-shell objects.



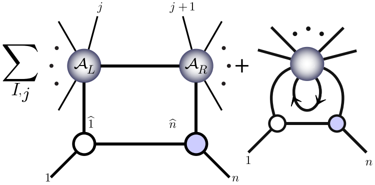
- On-shell diagrams in planar $\mathcal{N} = 4$ SYM
  - Cells in Positive Grassmannian $G_+(k, n)$.
  - Logarithmic form

$$\Omega_0 = \int \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d} \delta(C(x_i)Z_j)$$

Dual formulation

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT]

- BCFW recursion relation
  - Sum of on-shell diagrams.
  - Yangian invariant term-by-term.



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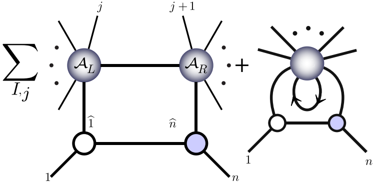

- **Amplituhedron** [Arkani-Hamed, JT]
  - Gluing on-shell diagrams together.
  - Geometric definition of the amplitude:
    - Defined by the set of inequalities.
    - Form with logarithmic singularities.

- Crucial property: Logarithmic singularities.

Logarithmic singularities

- Definition: Differential form $\Omega \sim \frac{dx}{x}\widetilde{\Omega}$ near $x = 0$.

$$\Omega = \frac{dx}{x} = \mathrm{dlog}\, x \quad \text{vs} \quad \Omega = \frac{dx}{x^2}$$

  - Multiple poles hidden in the cut structure

$$\Omega = \frac{dx\, dy}{xy(x+y)} \qquad \mathrm{Res}_{x=0}\Omega = \frac{dy}{y^2}$$

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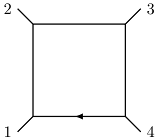
$$\Omega = \frac{dx\, dy}{xy(x+y)} \qquad \mathrm{Res}_{x=0}\Omega = \frac{dy}{y^2}$$

- **Dlog form**: $\Omega = \mathrm{dlog}\, f_1 \ \mathrm{dlog}\, f_2 \ldots \mathrm{dlog} f_m$

$$\Omega = \frac{dx\, dy\ (x-y)}{xy(x+y)} = \mathrm{dlog}\frac{x}{(x+y)}\ \mathrm{dlog}\frac{y}{(x+y)}$$

Singularities of loop integrals

- Example: box integral



$$I = \frac{d^4\ell \; st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

- Examples of integrals with non-logarithmic singularities:

$$I = \frac{d^4\ell}{(\ell^2)^2(\ell+k_1)^2(\ell+k_1+k_2)^2}, \quad I = \frac{d^4\ell}{\ell^2(\ell+k_1)^2(\ell+k_2)^2(\ell+k_3)^2}$$

- At higher loops: multiple poles
  $\rightarrow$ Special **numerator** needed to cancel them.

Poles at infinity

- We consider loop integrals.
- Type of singularities: logarithmic.
- Restriction on positions of singularities:

No poles for $\ell \to \infty$.

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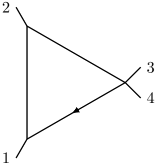
- Dual conformal symmetry
  - No infinity twistor $\to$ no poles at infinity.

Poles at infinity

- Example: triangle integral



$$I = \frac{d^4\ell \ s}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2}$$

  - Triple cut: $\ell^2 = (\ell+k_1)^2 = (\ell+k_1+k_2)^2 = 0$
  - Solution: $\ell - k_1 = \alpha\lambda_1\widetilde{\lambda_2}$

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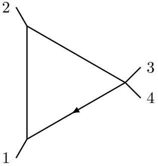

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  - Solution: $\ell - k_1 = \alpha\lambda_1\widetilde{\lambda_2}$
  - Residue on this cut: $$I = \frac{d\alpha}{\alpha}$$
  - Pole for $\alpha \to \infty$ which implies $\ell \to \infty$.

Planar amplitudes

- Dual formulation using on-shell diagrams:
  - No poles at infinity.
  - Logarithmic singularities in the Grassmannian space.
  - N^{k}MHV for $k = 0, 1, 2$: logarithmic singularities in **momentum space**.

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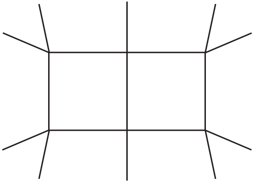
  - N^kMHV for $k = 0, 1, 2$: logarithmic singularities in **momentum space**.
  - Logarithmic singularities $\sim$ polylogs.
  - Non-logarithmic singularities in 10pt N^3MHV 2-loop: elliptic functions [Caron-Huot, Larsen]



Non-planar amplitudes

- Amplitudes in complete $\mathcal{N} = 4$ SYM
  - Still have: maximal supersymmetry, UV finiteness.
  - But: no DCI, no Yangian, no amplitudes/Wilson loop

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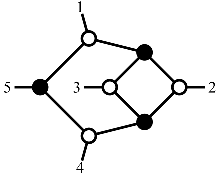
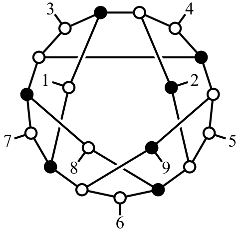
  - But: no DCI, no Yangian, no amplitudes/Wilson loop
- On-shell diagrams well defined





  - Even richer mathematical structure.
    [Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT]
  - Logarithmic singularities, no poles at infinity.

Non-planar amplitudes

- No dual formulation for amplitudes yet.
- Suppose we can find it!
- Natural conjecture:

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Amplitudes in complete $\mathcal{N}=4$ SYM have **logarithmic** singularities and **no poles at infinity**.

  - This is in the Grassmannian space.
  - In momentum space conservative conjecture: **four point**.

- No $1/N$ expansion, property of the full theory.

Non-planar amplitudes

- How to test the conjecture?
  - Analyze data up to 5-loops.
    [Bern, Carrasco, Dixon, Johansson, Kosower, Roiban]
  - Find the new basis with these two properties **manifest**.
  - Expand the amplitude in this basis.

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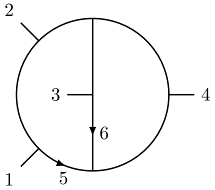
- Conservative strategy: scalar integrals.
  - Denominator given by the diagram.
  - Fix the **numerator**: cancels bad singularities.



Non-planar amplitudes

- Existence of such numerators not guaranteed.
  - Possible cancellations between diagrams.
- New expansion vs. reference
  - No unique integrand: no algebraic proof.
  - Match on cuts.

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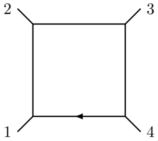
- Existence of such numerators not guaranteed.
  - Possible cancellations between diagrams.
- New expansion vs. reference
  - No unique integrand: no algebraic proof.
  - Match on cuts.
- Explicitly constructed and checked up to 3-loops.
- Few more checks at higher loops.

One-loop amplitude

- One-loop amplitude: sum over permutations of box



$$= \frac{d^4\ell \; st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

- Same structure as the planar amplitude.

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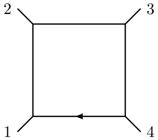

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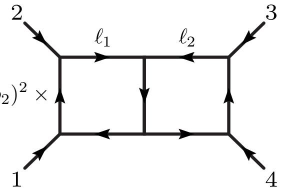
- Dlog form for box

$$\mathrm{dlog}\,\frac{\ell^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell+k_1)^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell+k_1+k_2)^2}{(\ell-\ell^*)^2}\;\mathrm{dlog}\,\frac{(\ell-k_4)^2}{(\ell-\ell^*)^2}$$

Two-loop amplitude

- Planar double-box

$$\mathcal{I}^{(P)}_{1,2,3,4} \equiv (p_1 + p_2)^2 \times$$



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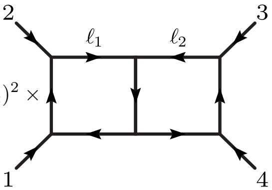

,

- ▸ written in the dlog form

$$st\mathcal{I}^{(P)}_{1234} = \mathrm{dlog}\,\alpha_1\ \mathrm{dlog}\,\alpha_2\ \mathrm{dlog}\,\alpha_3 \ldots \mathrm{dlog}\,\alpha_8$$

$$\begin{aligned}
\alpha_1 &\equiv \ell_1^2/(\ell_1 - \ell_1^*)^2, & \alpha_5 &\equiv \ell_2^2/(\ell_2 - \ell_2^*)^2,\\
\alpha_2 &\equiv (\ell_1 - p_2)^2/(\ell_1 - \ell_1^*)^2, & \alpha_6 &\equiv (\ell_1 + \ell_2)^2/(\ell_2 - \ell_2^*)^2,\\
\alpha_3 &\equiv (\ell_1 - p_1 - p_2)^2/(\ell_1 - \ell_1^*)^2, & \alpha_7 &\equiv (\ell_2 - p_3)^2/(\ell_2 - \ell_2^*)^2,\\
\alpha_4 &\equiv (\ell_1 + p_3)^2/(\ell_1 - \ell_1^*)^2, & \alpha_8 &\equiv (\ell_2 - p_3 - p_4)^2/(\ell_2 - \ell_2^*)^2,
\end{aligned}$$

Two-loop amplitude

- Non-planar double box

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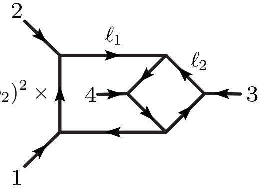

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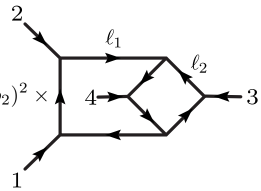

- Generate a **double pole**
  - Quadruple cut on ℓ_2.
  - Triple cut on $\ell_1 = x p_2$.

$$\text{Res } \mathcal{I} = \frac{dx}{(x+1)x^2 t u}$$

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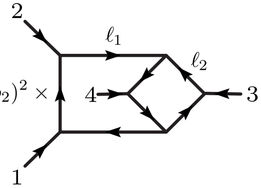

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- Numerator: $N_{old} = (p_1+p_2)^2 \to N_{new} = (\ell_1+p_3)^2 + (\ell_1+p_4)^2$
  - On the cut: $N_{new} = (xp_2+p_3)^2 + (xp_2+p_4)^2 = -xs$

Two-loop amplitude

- More double poles: N_{new} cancels all of them.
- Also conditions on the absence of the pole at infinity.
  - Everything resolved by N_{new}.

Two-loop amplitude

- More double poles: N_{new} cancels all of them.
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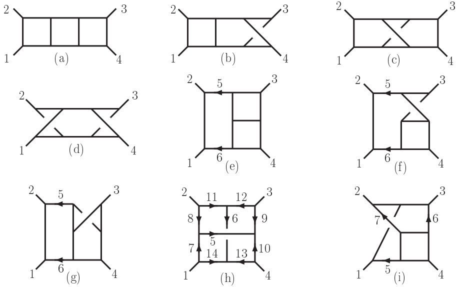
  - Everything resolved by N_{new}.
- Expand the amplitude in a new basis: YES.
- New result looks differently.
  - Difference cancels due to color **Jacobi identity**.

$$f^{abe} f^{cde} + f^{ace} f^{dbe} + f^{ade} f^{bce} = 0$$

Three-loop amplitude

- Nine diagrams in the basis.

[Bern, Carrasco, Dixon, Johansson, Kosower, Roiban, 2007]



- All except (a) double poles: **new numerators**.

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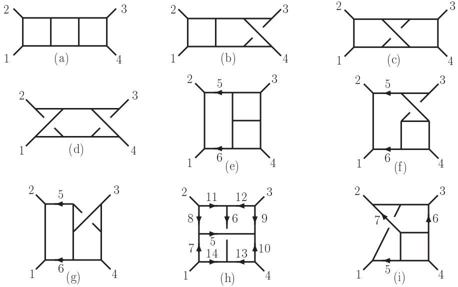


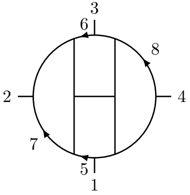
- All except (a) double poles: **new numerators**.

- Check completed: Amplitude expanded in new basis!

→ Talk by Enrico Herrmann

Higher loops

- Partial checks
  - New basis for some diagrams.
  - Match the reference on maximal cuts.

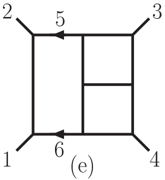


- All checks show that the conjecture is correct.

Back to planar sector

- In planar sector
  - Both properties manifest in the basis of on-shell diagrams.

- What about scalar integrals?
  - Do not consider DCI.
  - Follow only our two conditions.



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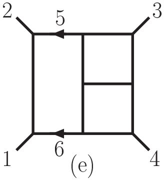

- Result:
  - In all cases we reproduce DCI numerators.
  - We get even **stronger** restrictions!

Beyond DCI

- Extra condition beyond DCI [Drummond, Korchemsky, Sokatchev]
  - Starting at 4-loops some DCI integrals **not well-defined**.
  - All dual loop momenta $x_5, x_6, \ldots$ go to external point.

$$\rho^2 = x_{35}^2 + x_{36}^2 + \cdots \to 0$$

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  - If the integral behaves like

$$I \sim \int \frac{d\rho}{\rho} \qquad \text{IR divergence even off-shell}$$

  - Zero coefficient in the amplitude.

Beyond DCI

- Equivalent to the presence of certain type of **multiple poles**!
- There are more rules of this type
  - Dual loop variables null separated from each other.

$$\rho^2 = x_{56}^2 + x_{57}^2 + \cdots \to 0$$

  - Null separated from two points

$$\rho^2 = x_{25}^2 + x_{35}^2 + x_{26}^2 + x_{36}^2 + \cdots \to 0$$

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- We have data up to 7-loops.

[Bern, Carrasco, Dixon, Johansson, Kosower, Smirnov; Bourjaily, DiRe, Shaikh, Spradlin, Volovich; Eden, Heslop, Korchemsky, Sokatchev]

- These rules explain zeroes in 4-loop and 5-loop expansions.

Beyond DCI

- Many more types of multiple poles.
- Requirement: All of them must be absent.
  - Explains many zeroes of DCI integrals at 6-loops and 7-loops.
  - Gives correct relative coefficients at 5-loops.

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    $\rightarrow$ Talk by Enrico Herrmann
- Cancellation of multiple poles at higher loops.
  - Possible in each diagram?
  - Between different diagrams?

Beyond DCI

- Multiple poles from [Drummond, Korchemsky, Sokatchev]
  - **Real momenta** on the solutions for these cuts.
  - Problem: they are in the domain of integration.

Beyond DCI

- Multiple poles from [Drummond, Korchemsky, Sokatchev]
  - **Real momenta** on the solutions for these cuts.
  - Problem: they are in the domain of integration.
- Other multiple poles in integrals:
  - Momenta are complex $\rightarrow$ outside integration domain.
  - Integral well-defined but non-uniform transcendentality.
- Conjecture for $\mathcal{N} = 4$ SYM: no multiple poles at all.

Beyond DCI

- Planar and non-planar integrals:
  - Fixed by the **same rules**.
- In planar sector:
  - Integrals with log singularities: automatically DCI.

Beyond DCI

- Planar and non-planar integrals:
  - Fixed by the **same rules**.
- In planar sector:
  - Integrals with log singularities: automatically DCI.
- It suggests the existence of the non-planar analogue of DCI!

Beyond DCI

- Planar sector: logarithmic singularities and no poles at infinity

$$I = \sum_j \alpha_j (\mathrm{dlog} \ldots \mathrm{dlog})_j$$

Beyond DCI

- Planar sector: logarithmic singularities and no poles at infinity

$$I = \sum_j \alpha_j (\mathrm{dlog} \ldots \mathrm{dlog})_j$$

- Two things can go wrong

1. Arguments of dlogs are non-DCI (without poles at infinity).
2. Coefficients α_j are non-DCI (mod overall constant).

- Explicit data: None of it happens.

Supergravity

- Extend ideas to other theories:
  - Lower supersymmetry: new singularity structure.
  - From Yang-Mills to **gravity**.

Supergravity

- Extend ideas to other theories:
  - Lower supersymmetry: new singularity structure.
  - From Yang-Mills to **gravity**.
- Maximal $\mathcal{N} = 8$ SUGRA
  - Singularity structure and UV behavior unclear.
  - Close relation to $\mathcal{N} = 4$ SYM via BCJ relations. [Bern, Carrasco, Johansson]
  - Natural idea: explore singularity structure of the integrand.
- At 2-loops logarithmic singularities and no poles at infinity.

Supergravity

- No poles at infinity $\rightarrow$ **UV finiteness**.
- For $\mathcal{N} = 4$ SYM: integrand-based derivation of UV finiteness.
- If true for $\mathcal{N} = 8$ SYM: trivially UV finite as well.

Supergravity

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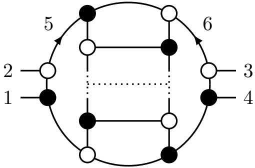
- If true for $\mathcal{N} = 8$ SYM: trivially UV finite as well.
- Explicit checks of poles at infinity
  - No poles at 1-loop and 2-loops.
  - Logarithmic at 3-loops.
  - Non-logarithmic at 4-loops, ...



- Results: Poles at infinity **are present**.

Supergravity

- This is **NOT** a proof of UV divergence.
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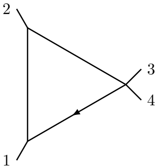

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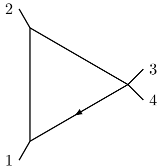

- Possible finiteness of $\mathcal{N} = 8$ SUGRA:
  - Must depend on **detailed structures** of poles at infinity.
  - More complicated than $\mathcal{N} = 4$ SYM.
  - Cancellations between diagrams $\rightarrow$ Talk by Scott Davies.

Conclusion

- Amplitudes in complete $\mathcal{N} = 4$ SYM: singularities of integrand.

- Inspired by properties of on-shell diagrams:

<u>Conjecture</u>: Four point amplitudes in $\mathcal{N} = 4$ SYM have only logarithmic singularities and no poles at infinity.

- Explicit checks up to 3-loops.

- In planar sector: stronger conditions than DCI.

- $\mathcal{N} = 8$ SUGRA: poles at infinity

$\rightarrow$ Detailed knowledge needed to study UV behavior.

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Thank you for the attention!