Functions associated to scattering amplitudes

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I: **Periodic functions and periods**

II: **Differential equations**

III: **The two-loop sun-rise diagramm**

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Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) \quad = \quad f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{ n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z} \},$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i \quad = \quad 2i \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2\int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad \omega_2 = 2\int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3 - g_2 t - g_3 = 0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace “rational” with “algebraic”.
- The set of all periods is countable.
- Example: $\ln 2$ is a numerical period.

$$\ln 2 \quad = \quad \int_1^2 \frac{dt}{t}.$$

Feynman integrals

A Feynman graph with m external lines, n internal lines and l loops corresponds (up to prefactors) in D space-time dimensions to the Feynman integral

$$I_G = \frac{\left(\mu^2\right)^{n-lD/2}}{\Gamma(n-lD/2)} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{(-q_j^2+m_j^2)}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta k_1, ..., k_l and the external momenta p_1, ..., p_m as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{m} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

Feynman parametrisation

The Feynman trick:

$$\prod_{j=1}^{n} \frac{1}{P_j} = \Gamma(n) \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{j=1}^{n} x_j) \, \frac{1}{\left(\sum\limits_{j=1}^{n} x_j P_j \right)^n}$$

We use this formula with $P_j = -q_j^2 + m_j^2$.
We can write

$$\sum_{j=1}^{n} x_j (-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r M_{rs} k_s + \sum_{r=1}^{l} 2 k_r \cdot Q_r + J,$$

where M is a $l \times l$ matrix with scalar entries and Q is a l-vector with momenta vectors as entries.

Feynman integrals

After Feynman parametrisation the integrals over the loop momenta k_1, ..., k_l can be done:

$$I_G = \int\limits_{x_j \geq 0} d^n x \, \delta(1 - \sum_{i=1}^{n} x_i) \frac{\mathcal{U}^{n-(l+1)D/2}}{\mathcal{F}^{n-lD/2}}, \qquad \mathcal{U} = \det(M),$$

$$\mathcal{F} = \det(M) \left(J + Q M^{-1} Q \right) / \mu^2.$$

The functions $\mathcal{U}$ and $\mathcal{F}$ are called the first and second graph polynomial.

$\mathcal{U}$ is positive definite inside the integration region and positive semi-definite on the boundary.

$\mathcal{F}$ depends on the masses m_i^2 and the momenta $(p_{i_1} + ... + p_{i_r})^2$. In the euclidean region $\mathcal{F}$ is also positive definite inside the integration region and positive semi-definite on the boundary.

Feynman integrals and periods

Laurent expansion in $\varepsilon = (4-D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Question: What can be said about the coefficients c_j ?

Theorem: For rational input data in the euclidean region the coefficients c_j of the Laurent expansion are numerical periods.

(Bogner, S.W., '07)

Next question: Which periods ?

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\ \text{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2} \end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms (Goncharov 1998):

$$\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k) = \sum_{n_1>n_2>...>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot ... \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

$$... \sum_{n_j=1}^{n_{j-1}-1} \frac{x_j^{n_j}}{n_j^{m_j}} \sum_{n_{j+1}=1}^{n_j-1} ...$$

Iterated integrals

Define the functions G by

$$G(z_1,...,z_k;y) = \int\limits_0^y \frac{dt_1}{t_1 - z_1} \int\limits_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int\limits_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}.$$

Scaling relation:

$$G(z_1,...,z_k;y) = G(xz_1,...,xz_k;xy)$$

Short hand notation:

$$G_{m_1,...,m_k}(z_1,...,z_k;y) = G(\underbrace{0,...,0}_{m_1-1},z_1,...,z_{k-1},\underbrace{0...,0}_{m_k-1},z_k;y)$$

Conversion to multiple polylogarithms:

$$\text{Li}_{m_1,...,m_k}(x_1,...,x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1},\frac{1}{x_1 x_2},...,\frac{1}{x_1 ... x_k};1\right).$$

Differential equations for Feynman integrals

If it is not feasible to compute the integral directly:

Pick one variable t from the set s_{jk} and m_i^2.

1. Find a differential equation for the Feynman integral.

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} I_G(t) \quad = \quad \sum_i q_i(t) I_{G_i}(t)$$

   Inhomogeneous term on the rhs consists of simpler integrals I_{G_i}.

   $p_j(t)$, $q_i(t)$ polynomials in t.

2. Solve the differential equation.

Kotikov; Remiddi, Gehrmann; Laporta; Argeri, Mastrolia, S. Müller-Stach, S.W., R. Zayadeh; Henn; ...

Differential equations: The case of multiple polylogarithms

Suppose the differential operator factorises into linear factors:

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} = \left(a_r(t) \frac{d}{dt} + b_r(t) \right) \dots \left(a_2(t) \frac{d}{dt} + b_2(t) \right) \left(a_1(t) \frac{d}{dt} + b_1(t) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(t) = \exp\left(- \int\limits_0^t ds \frac{b_j(s)}{a_j(s)} \right).$$

Full solution given by iterated integrals

$$I_G(t) = C_1 \psi_1(t) + C_2 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} + C_3 \psi_1(t) \int\limits_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} \int\limits_0^{t_1} dt_2 \frac{\psi_3(t_2)}{a_2(t_2)\psi_2(t_2)} + \dots$$

Multiple polylogarithms are of this form.

Differential equations: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(t) \frac{d^2}{dt^2} + b_j(t) \frac{d}{dt} + c_j(t)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[t\left(1-t^2\right)\frac{d^2}{dt^2}+\left(1-3t^2\right)\frac{d}{dt}-t\right]f(t) \quad = \quad 0$$

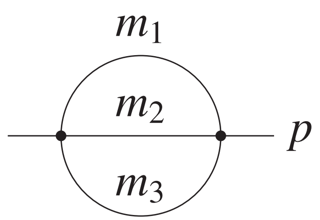
is irreducible.

The solutions of the differential equation are $K(t)$ and $K(\sqrt{1-t^2})$, where $K(t)$ is the complete elliptic integral of the first kind:

$$K(t) \quad = \quad \int_0^1 \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-t^2x^2\right)}}.$$

An example from physics: The two-loop sunrise integral

$$S\left(p^2, m_1^2, m_2^2, m_3^2\right) \quad =$$



- Two-loop contribution to the self-energy of massive particles.
- Sub-topology for more complicated diagrams.

The two-loop sunrise integral: Prior art

Integration-by-parts identities allow to derive a coupled system of 4 first-order differential equations for S and S_1, S_2, S_3, where

$$S_i = \frac{\partial}{\partial m_i^2} S$$

(Caffo, Czyz, Laporta, Remiddi, 1998).

This system reduces to a single second-order differential equation in the case of equal masses $m_1 = m_2 = m_3$

(Broadhurst, Fleischer, Tarasov, 1993).

Dimensional recurrence relations relate integrals in $D = 4$ dimensions and $D = 2$ dimensions

(Tarasov, 1996, Baikov, 1997, Lee, 2010).

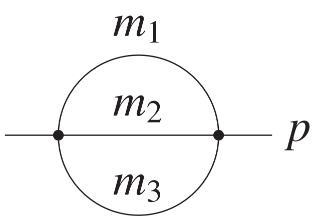
Analytic result in the equal mass case known up to quadrature, result involves elliptic integrals

(Laporta, Remiddi, 2004).

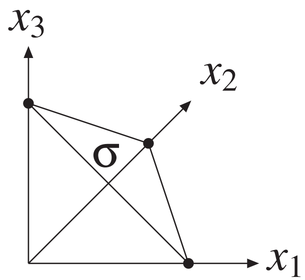
The two-loop sunrise integral in two dimensions

The two-loop sunrise integral with non-zero masses in two-dimensions ($t = p^2$):

$$S(t) =$$



$$= \int_\sigma \frac{\omega}{\mathcal{F}},$$



$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2,$$

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right)$$

Algebraic geometry studies the zero sets of polynomials.

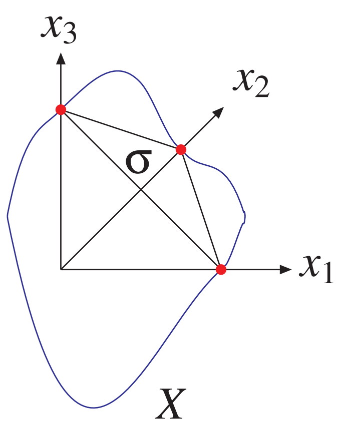
In this case look at the set $\mathcal{F} = 0$.

The two-loop sunrise integral

From the point of view of algebraic geometry there are two objects of interest:

- the domain of integration σ,
- the zero set X of $\mathcal{F} = 0$.

X and σ intersect at three points:



The elliptic curve

Algebraic variety X defined by the polynomial in the denominator:

$$-x_1x_2x_3t + \left(x_1m_1^2 + x_2m_2^2 + x_3m_3^2\right)\left(x_1x_2 + x_2x_3 + x_3x_1\right) = 0.$$

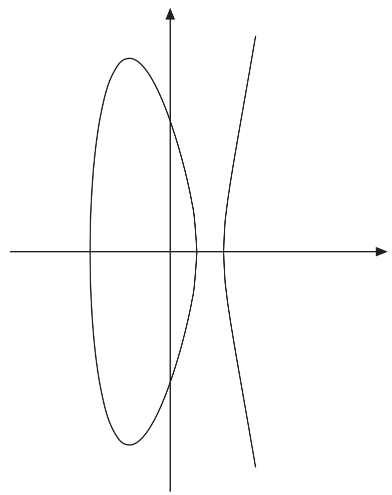
This defines (together with a choice of a rational point as origin) an elliptic curve.

Change of coordinates $\rightarrow$ Weierstrass normal form

$$y^2z - 4x^3 + g_2(t)xz^2 + g_3(t)z^3 = 0.$$

In the chart $z = 1$ this reduces to

$$y^2 - 4x^3 + g_2(t)x + g_3(t) = 0.$$



$y^2 = 4x^3 - 28x + 24$

The curve varies with t.

Abstract periods

Input:

- X a smooth algebraic variety of dimension n defined over $\mathbb{Q}$,
- $D \subset X$ a divisor with normal crossings (i.e. a subvariety of dimension $n-1$, which looks locally like a union of coordinate hyperplanes),
- ω an algebraic differential form on X of degree n,
- σ a singular n-chain on the complex manifold $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$.

To each quadruple (X, D, ω, σ) associate the period

$$P(X, D, \omega, \sigma) \quad = \quad \int_\sigma \omega.$$

The motive

P: Blow-up of $\mathbb{P}^2$ in the three points, where X intersects σ.

Y: Strict transform of the zero set X of $\mathcal{F} = 0$.

B: Total transform of $\{x_1 x_2 x_3 = 0\}$.

Mixed Hodge structure:

$$H^2(P \backslash Y, B \backslash B \cap Y)$$

(S. Bloch, H. Esnault, D. Kreimer, 2006)

We need to analyse $H^2(P \backslash Y, B \backslash B \cap Y)$.
We can show that essential information is given by $H^1(X)$.

(S. Müller-Stach, S.W., R. Zayadeh, 2011)

The second-order differential equation

In the Weierstrass normal form $H^1(X)$ is generated by

$$\eta = \frac{dx}{y} \quad \text{and} \quad \dot{\eta} = \frac{d}{dt}\eta.$$

$\ddot{\eta} = \frac{d^2}{dt^2}\eta$ must be a linear combination of η and $\dot{\eta}$:

$$p_0(t)\ddot{\eta} + p_1(t)\dot{\eta} + p_2(t)\eta \quad = \quad 0.$$

Differential equation:

$$\left[p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)\right] S(t) \quad = \quad p_3(t)$$

p_0, p_1, p_2 and p_3 are polynomials in t.

Periods of an elliptic curve

In the Weierstrass normal form, factorise the cubic polynomial in x:

$$y^2 = 4\left(x-e_1\right)\left(x-e_2\right)\left(x-e_3\right).$$

Holomorphic one-form is $\frac{dx}{y}$, associated periods are

$$\psi_1\left(t\right) = 2\int\limits_{e_2}^{e_3} \frac{dx}{y}, \qquad \psi_2\left(t\right) = 2\int\limits_{e_1}^{e_3} \frac{dx}{y}.$$

These periods are the solutions of the homogeneous differential equation.

L. Adams, Ch. Bogner, S.W., '13

The full result

- Once the homogeneous solutions are known, variation of the constants yields the full result up to quadrature:
  - Equal mass case: Laporta, Remiddi, '04
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., '13
- The full result can be expressed in terms of elliptic dilogarithms:
  - Equal mass case: Bloch, Vanhove, '13
  - Unequal mass case: L. Adams, Ch. Bogner, S.W., '14

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) = \frac{1}{i} \left[\frac{1}{2} \mathrm{Li}_2(x) - \frac{1}{2} \mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right) \right].$$

(Slightly) different definitions of elliptic polylogarithms can be found in the literature

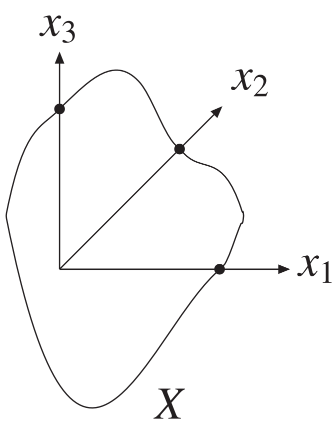
Beilinson '94, Levin '97, Brown, Levin '11, Wildeshaus '97.

Elliptic curves again

The nome q is given by

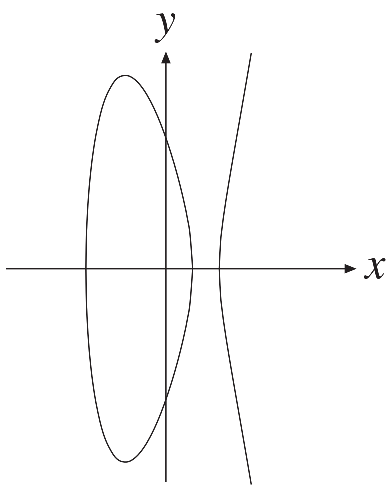
$$q = e^{i\pi\tau} \quad \text{with} \quad \tau = \frac{\psi_2}{\psi_1} = i\frac{K(k')}{K(k)}.$$

Elliptic curve represented by



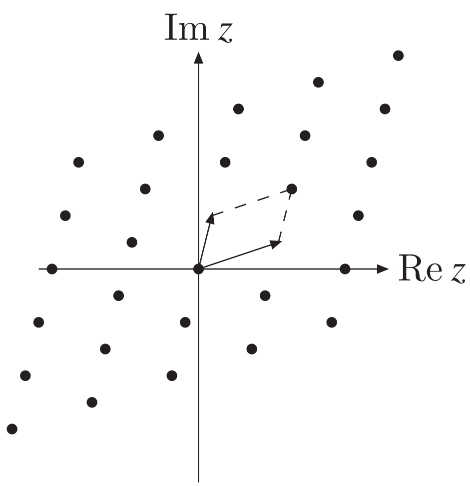
Algebraic variety

$\mathcal{F} = 0$



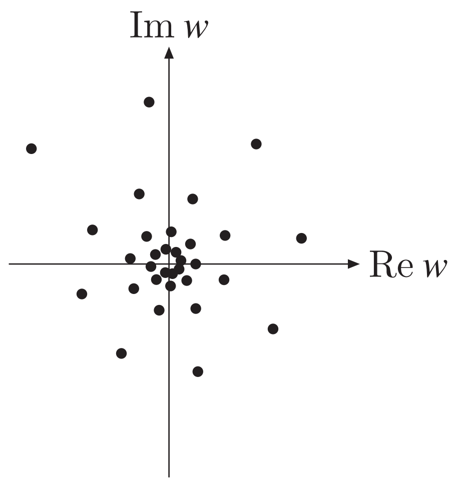
Weierstrass normal form

$y^2 = 4x^3 - g_2 x - g_3$



Torus

$\mathbb{C}/\Lambda$



Jacobi uniformization

$\mathbb{C}^*/q^{2\mathbb{Z}}$

The arguments of the elliptic dilogarithms

Elliptic curve: Cubic curve together with a choice of a rational point as the origin O.

Distinguished points are the points on the intersection of the cubic curve $\mathcal{F} = 0$ with the domain of integration σ:

$$P_1 = [1:0:0], \quad P_2 = [0:1:0], \quad P_3 = [0:0:1].$$

Choose one of these three points as origin and look at the image of the two other points in the Jacobi uniformization $\mathbb{C}^*/q^{2\mathbb{Z}}$ of the elliptic curve. Repeat for the two other choices of the origin. This defines

$$w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}.$$

In other words: $w_1, w_2, w_3, w_1^{-1}, w_2^{-1}, w_3^{-1}$ are the images of P_1, P_2, P_3 under

$$E_i \longrightarrow \mathrm{WNF} \longrightarrow \mathbb{C}/\Lambda \longrightarrow \mathbb{C}^*/q^{2\mathbb{Z}}.$$

The full result in terms of elliptic dilogarithms

The result for the two-loop sunrise integral in two space-time dimensions with arbitrary masses:

$$S = \underbrace{\frac{4}{\left[\left(t-\mu_1^2\right)\left(t-\mu_2^2\right)\left(t-\mu_3^2\right)\left(t-\mu_4^2\right)\right]^{\frac{1}{4}}}}_{\text{algebraic prefactor}} \quad \underbrace{\frac{K(k)}{\pi}}_{\text{elliptic integral}} \quad \underbrace{\sum_{j=1}^{3} \mathrm{E}_{2;0}\left(w_j;-1;-q\right)}_{\text{elliptic dilogarithms}}$$

t	momentum squared
μ_1,μ_2,μ_3	pseudo-thresholds
μ_4	threshold
$K(k)$	complete elliptic integrals of the first kind
k,q	modulus and nome
w_1,w_2,w_3	points in the Jacobi uniformization

Conclusions

Question: What is the next level of sophistication beyond multiple polylogarithms for Feynman integrals?

Answer: Elliptic stuff.

- Algebraic prefactors as before.
- Elliptic integrals generalise the period π.
- Elliptic (multiple) polylogarithms generalise the (multiple) polylogarithms.
- Arguments of the elliptic polylogarithms are points in the Jacobi uniformization of the elliptic curve.

Opportunities at Mainz University

International Ph.D. program in fundamental physics:

- Graduate School 'Symmetry Breaking in Fundamental Interactions'

  `http://www.symmetrybreaking.uni-mainz.de/`

- Cluster of Excellence 'Precision Physics, Fundamental Interactions and Structure of Matter (PRISMA)'

  `http://www.prisma.uni-mainz.de/`