Generalized BCFW Recursion Relation with Boundary Terms

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Nov 20, 2014

BCFW Recursion Relation

- The formula [Britto, Cachazo, Feng & Witten, '05] :

$$A = \sum_{\alpha} A_L(\hat{z}_\alpha) \frac{1}{p_\alpha^2} A_R(\hat{z}_\alpha). \tag{1}$$

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- Its derivation: Deform a spinor pair $\langle i|j]$ as

$$|i\rangle \to |i\rangle + z|j\rangle, \ \ |j] \to |j] - z|i], \tag{2}$$

$$p_\alpha^2 \to p_\alpha^2 - z\langle i|p_\alpha|j], \ \ i \in p_\alpha, \ \ j \notin p_\alpha. \tag{3}$$

All momenta remain null and momentum conservation still holds.

$$\oint_0 \frac{dz}{z} A(z) + \sum_{p_\alpha^2=0} \oint \frac{dz}{z} A(z) = \oint_\infty \frac{dz}{z} A(z) = B. \tag{4}$$

- Good deformations do not have boundary terms.

$$\lim_{z\to\infty} A(z) = 0 \Longrightarrow B = 0. \tag{5}$$

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- By applying the background field method, when the amplitude contains at least one gluon or graviton, there always exists one good deformation [N. Arkani-Hamed & J. Kaplan, '08; C. Cheung, '08] .

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- What about interaction theories of scalars and fermions, *i.e.*, the Yukawa theory?
- When it is unavoidable to encounter boundary terms, how can one calculate them, and what information do they imply?

Proposals to Handle Boundary Terms

- (1) Add auxiliary fields to heal the large z behavior
  [P. Benincasa & F. Cachazo, '07; R. H. Boels, '10]
  It is unknown in general whether the enlarged theory exists, or how to construct it if it exists. And the parental amplitudes could be far more complicated than expected.

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- (2) Analyze Feynman diagrams to isolate their boundary terms
  [B. Feng, J. Wang, Y. Wang & Z. Zhang, '09; B. Feng & C. Y. Liu, '10; B. Feng & Z. Zhang, '11]
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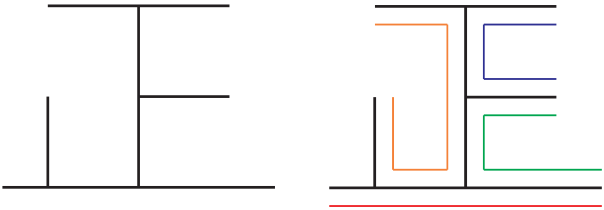
  It is extremely challenging to find roots.
- A universal, systematic approach is still lacking.

The New Algorithm

More careful analysis: What's wrong with bad deformations?

- A deformation is an injection of large z, so it may have the 'short-circuit' problem: the large complex momentum flow misses part of all physical channels.



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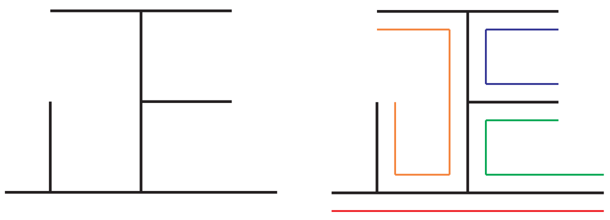

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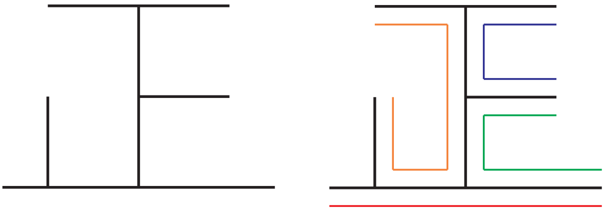
- Multiple deformations have overlapping detected channels, how to deal with the overcounting?
- Two-particle channels are special: They factorize into holomorphic and anti-holomorphic parts, *i.e.*, $p_{ij}^2 = \langle ij\rangle[ij]$.



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- More symbolic abstraction: Extraction operator formalism
  [JR, 2 weeks ago]

Extraction Operator Formalism

- For a general BCFW deformation $\langle i_t | j_t]$ which detects physical poles in the set D^t, define

$$P^t A \equiv - \sum_{\text{pole} \in \mathcal{D}^t} \oint \frac{dz_t}{z_t} A(z_t), \; C^t A \equiv \oint_\infty \frac{dz_t}{z_t} A(z_t), \qquad (6)$$

P^t and C^t are the pole and constant extraction operators respectively, with $P^t + C^t = I^t$. In the beginning C^0 is unknown, but P^0 represents exactly the BCFW recursion relation.
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- Expanding I^i repeatedly, yields

$$I^{10} = P^1 + C^1 I^0 = P^1 + C^1 P^0 + C^{10}, \quad (7)$$

$$I^{210} = P^2 + C^2 I^{10} = P^2 + C^2 P^1 + C^{21} P^0 + C^{210}, \quad (8)$$

$$I^{3210} = P^3 + C^3 I^{210} = P^3 + C^3 P^2 + C^{32} P^1 + C^{321} P^0 + C^{3210}. \quad (9)$$

- If I^{3210} covers all possible physical poles and the final expression is free of spurious poles, we can set $C^{3210} = 0$, hence

$$I^{3210} = P^3 + C^3 P^2 + C^{32} P^1 + C^{321} P^0. \tag{10}$$

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- This algorithm is called the CP scheme, where at each step we extract the constant part from the pole part of last step accumulatively. Another equivalent representation is the PC scheme:

$$I^{10} = P^0 + I^1 C^0 = P^0 + P^1 C^0 + C^{10}, \tag{11}$$

$$P^1 C^0 = P^1_{\mathcal{D}^1 \setminus \mathcal{D}^1 \cap \mathcal{D}^0} (I - P^0). \tag{12}$$

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- We should only consider new poles because each physical pole only appears once. This is more than the 'Schmidt orthogonalization'.

- More steps of expansions:

$$I^{210} = P^0 + P^1C^0 + P^2C^{10} + C^{210}, \tag{13}$$

$$P^2C^{10} = P^2_{\mathcal{D}^2\backslash\mathcal{D}^2\cap(\mathcal{D}^1\cup\mathcal{D}^0)}(I - P^0 - P^1C^0), \tag{14}$$

and

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- Visually at each step we extract the pole part from the constant part of last step accumulatively, but in fact we express all constant extractions in terms of pole extractions.
- If a spurious pole appears, the same additional I^s is used to kill it.

Algebraic properties: Projectivity and commutativity

- It is easier to manipulate C^i's, and P^i's follow from $P^i = I - C^i$.

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- Performing the same deformation twice is equivalent to replacing z_i by $(z_i + z_i')$, hence the constant term remains the same.
  From a dual expansion of A in both z_i and z_j, the constant term is with respect to both variables, hence the order to perform two trivial contour integrals around infinity is irrelevant.

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- From above one can derive

$$P^iP^i = P^i,\ C^iP^i = P^iC^i = 0,\ P^iP^j = P^jP^i,\ C^iP^j = P^jC^i. \tag{18}$$

  Then each term in an expansion of either PC or CP scheme is 'orthogonal' to the others. But we already have a stronger condition: Each physical pole only appears once.

Tricks in practical applications

- Assume $I^{3210} = (\ldots) + C^{3210}$ is obtained, we can further expand it either as (PC scheme)

$$I^{43210} = (\ldots) + P^4 C^{3210} + C^{43210}, \tag{19}$$

or (CP scheme)

$$I^{43210} = P^4 + C^4(\ldots) + C^{43210}. \tag{20}$$

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- The 'last good deformation' corollary in the CP scheme:
In above if I^{43210} covers all physical poles, then if

$$C^4(\ldots) = 0, \tag{21}$$

step 4 must be a good deformation.

A Little Digression

C versus P: A consistency check (analytically)

- For an n-point amplitude, all $\langle ij \rangle$'s are:

$$\begin{array}{cccccc} \langle 12 \rangle & & & & & \\ \langle 13 \rangle & \langle 23 \rangle & & & & \\ \langle 14 \rangle & \langle 24 \rangle & \langle 34 \rangle & & & \\ \langle 15 \rangle & \langle 25 \rangle & \langle 35 \rangle & \langle 45 \rangle & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \langle 1n \rangle & \langle 2n \rangle & \langle 3n \rangle & \langle 4n \rangle & \ldots & \langle n-1, n \rangle \end{array} \tag{22}$$

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- For $i,j\neq 1,2$ one can solve all independent Schouten identities via

$$\langle ij\rangle = \frac{1}{\langle 12\rangle}\begin{vmatrix} \langle 1i\rangle & \langle 2i\rangle \\ \langle 1j\rangle & \langle 2j\rangle \end{vmatrix}. \tag{23}$$

There are C_{n-2}^2 Schouten identities, hence so far there are $C_n^2 - C_{n-2}^2 = 2n-3$ independent $\langle ij\rangle$'s left.

- Same for $[ij]$'s:

$$\begin{array}{cc} \langle 12\rangle & \\ \langle 13\rangle & \langle 23\rangle \\ \vdots & \vdots \\ \langle 1n\rangle & \langle 2n\rangle \end{array} \qquad \begin{array}{cc} [12] & \\ [13] & [23] \\ \vdots & \vdots \\ [1n] & [2n] \end{array} \tag{24}$$

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- Momentum conservation can solve four variables via

$$\begin{aligned} \langle 1|\sum P|1] = 0: &\quad \langle 13\rangle[13] + \langle 14\rangle[14] = -\Sigma_{11} - \langle 12\rangle[12], \\ \langle 2|\sum P|1] = 0: &\quad \langle 23\rangle[13] + \langle 24\rangle[14] = -\Sigma_{21}, \\ \langle 1|\sum P|2] = 0: &\quad \langle 13\rangle[23] + \langle 14\rangle[24] = -\Sigma_{12}, \\ \langle 2|\sum P|2] = 0: &\quad \langle 23\rangle[23] + \langle 24\rangle[24] = -\Sigma_{22} - \langle 12\rangle[12], \end{aligned} \tag{25}$$

where $\Sigma_{ij} = \sum_{k=5}^{n}\langle ik\rangle[jk]$. Now only $(4n-10)$ independent kinematic variables are left.

Consistency check

```
(V[2, 1] V[5, 3]
   (-(V[3, 1]^3 V[4, 2]^2 (6 L[1, 3] L[2, 5] V[2, 1] V[3, 2] V[4, 1] +
          V[4, 2] (-4 L[2, 5] L[3, 4] V[3, 2] V[4, 1] + L[2, 3] V[3, 1] (4 L[1, 5] V[2, 1] + L[4, 5] V[4, 2]))) +
        2 L[2, 5] V[2, 1] V[3, 1]^2 V[3, 2] V[4, 1] V[4, 2] (-2 L[1, 3] V[2, 1] + 3 L[3, 4] V[4, 2]) V[4, 3] +
        L[2, 5] V[2, 1]^2 V[3, 1] V[3, 2] V[4, 1] (L[1, 3] V[2, 1] - 4 L[3, 4] V[4, 2]) V[4, 3]^2 + L[2, 5] L[3, 4] V[2, 1]^3 V[3, 2] V[4, 1] V[4, 3]^3)
     V[5, 1]^4 +
    V[2, 1] V[3, 1]
     (-2 V[3, 1]^2 V[4, 2] (2 L[1, 3] L[2, 5] V[2, 1] V[3, 2] V[4, 1] +
          V[4, 2] (-3 L[2, 5] L[3, 4] V[3, 2] V[4, 1] + L[2, 3] V[3, 1] (3 L[1, 5] V[2, 1] + 2 L[4, 5] V[4, 2]))) +
       L[2, 5] V[2, 1] V[3, 1] V[3, 2] V[4, 1] (L[1, 3] V[2, 1] - 4 L[3, 4] V[4, 2]) V[4, 3] + L[2, 5] L[3, 4] V[2, 1]^2 V[3, 2] V[4, 1] V[4, 3]^2)
     V[5, 1]^3 V[5, 4] -
    V[2, 1]^2 V[3, 1]^2
     (V[3, 1] (L[1, 3] L[2, 5] V[2, 1] V[3, 2] V[4, 1] +
          2 V[4, 2] (-2 L[2, 5] L[3, 4] V[3, 2] V[4, 1] + L[2, 3] V[3, 1] (2 L[1, 5] V[2, 1] + 3 L[4, 5] V[4, 2]))) +
       L[2, 5] L[3, 4] V[2, 1] V[3, 2] V[4, 1] V[4, 3]) V[5, 1]^2 V[5, 4]^2 +
    V[2, 1]^3 V[3, 1]^3 (L[2, 5] L[3, 4] V[3, 2] V[4, 1] - L[2, 3] V[3, 1] (L[1, 5] V[2, 1] + 4 L[4, 5] V[4, 2])) V[5, 1] V[5, 4]^3 -
    L[2, 3] L[4, 5] V[2, 1]^4 V[3, 1]^4 V[5, 4]^4) -
  L[2, 4] L[3, 5] V[3, 1] V[3, 2] V[5, 1]
   (-V[2, 1]^3 V[4, 1]^2 V[4, 3]^2 V[5, 1]^2 V[5, 2] V[5, 3] + 5 V[3, 1]^3 (V[4, 2] V[5, 1] + V[2, 1] V[5, 4])^4 +
    2 V[2, 1]^2 V[3, 1] V[4, 1]^2 V[4, 3] V[5, 1] V[5, 2] V[5, 3] (2 V[4, 2] V[5, 1] + V[2, 1] V[5, 4]) -
    V[2, 1] V[3, 1]^2 V[4, 1]^2 V[5, 2] V[5, 3] (6 V[4, 2]^2 V[5, 1]^2 + 8 V[2, 1] V[4, 2] V[5, 1] V[5, 4] + 3 V[2, 1]^2 V[5, 4]^2)) +
  L[1, 2] L[3, 5] V[2, 1] V[3, 1]^2 V[3, 2] V[5, 1]^2
   (V[2, 1]^2 V[4, 1]^2 V[4, 3] V[5, 1] V[5, 2] V[5, 3] - 2 V[2, 1] V[3, 1] V[4, 1]^2 V[5, 2] V[5, 3] (2 V[4, 2] V[5, 1] + V[2, 1] V[5, 4]) +
    V[3, 1]^2 (10 V[4, 2]^3 V[5, 1]^3 + 20 V[2, 1] V[4, 2]^2 V[5, 1]^2 V[5, 4] + 15 V[2, 1]^2 V[4, 2] V[5, 1] V[5, 4]^2 + 4 V[2, 1]^3 V[5, 4]^3))) //
 FullSimplify
```

```
V[i,j]=<ij> L[i,j]=[ij]
V[1, 2] = x;
V[1, 3] = a;
V[1, 4] = c;
V[1, 5] = e;
V[2, 3] = b;
V[2, 4] = d;
V[2, 5] = f;
L[1, 2] = y;
L[1, 5] = g;
L[2, 5] = h;
```

$$\frac{1}{(b c - a d)^2} x \left(b^6 c^5 e^5 h (e g + f h + x y) - a^5 b h \left(d^5 e^6 g + c^5 f^6 h + (d e + c f) \left(d^2 e^2 - c d e f + c^2 f^2\right) \left(d^2 e^2 + c d e f + c^2 f^2\right) x y\right) + a^6 f \left(d^4 e^4 + c d^3 e^3 f + c^2 d^2 e^2 f^2 + c^3 d e f^3 + c^4 f^4\right) g (d e h - c (f h + x y))\right)$$

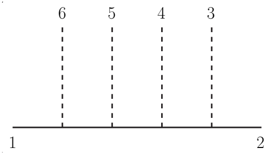
```
V[3, 1]^5 V[5, 1]^5 V[4, 2]^5 / (V[2, 1] V[5, 4] V[4, 3]) (V[1, 5] V[3, 4] L[1, 4] L[5, 3] - V[1, 4] V[5, 3] L[1, 5] L[3, 4]) +
 L[2, 5] L[3, 4] V[3, 2]^6 V[4, 1]^5 V[5, 1]^5 / (V[2, 1] V[4, 3]) - L[2, 3] L[4, 5] V[3, 1]^5 V[4, 1]^5 V[5, 2]^6 / (V[2, 1] V[5, 4]) // FullSimplify
```

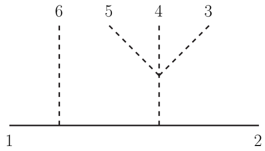
$$\frac{1}{(b c - a d)^2} x \left(b^6 c^5 e^5 h (e g + f h + x y) - a^5 b h \left(d^5 e^6 g + c^5 f^6 h + (d e + c f) \left(d^2 e^2 - c d e f + c^2 f^2\right) \left(d^2 e^2 + c d e f + c^2 f^2\right) x y\right) + a^6 f \left(d^4 e^4 + c d^3 e^3 f + c^2 d^2 e^2 f^2 + c^3 d e f^3 + c^4 f^4\right) g (d e h - c (f h + x y))\right)$$

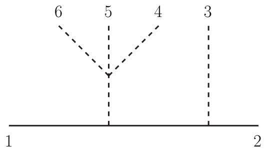
Applications

Yukawa with ϕ^4

- Color-ordered amplitude $A(1^-, 2^+, 3, 4, 5, 6)$: 1^- and 2^+ are fermions and the rest are scalars. There are Yukawa coupling g and ϕ^4 coupling λ.







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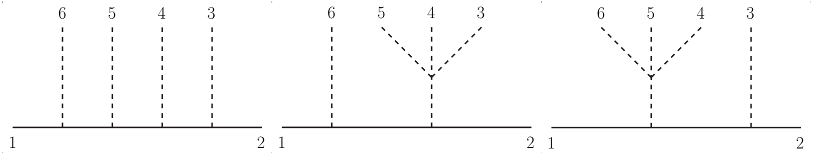

- Observe that: $\langle 5|6]$ deformation detects $[16]$ and s_{345}, $\langle 3|4]$ deformation detects $\langle 23\rangle$ and s_{456}, $\langle 4|5]$ deformation detects s_{234}. $\langle 16\rangle$ and $[23]$ are excluded by the helicity configuration $(1^-, 2^+)$.

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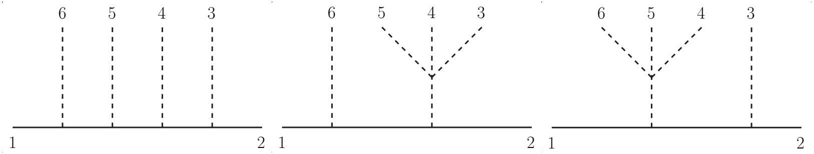

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- These are all bad BCFW deformations.

- In the CP scheme,

$$A = P^{\langle 4|5]} A + C^{\langle 4|5]} P^{\langle 3|4]} A + C^{\langle 4|5]\langle 3|4]} P^{\langle 5|6]} A, \tag{26}$$

where $C^{\langle 4|5]\langle 3|4]\langle 5|6]} A = 0$, if there is no unexpected spurious pole.

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- Start with ordinary BCFW recursion for $\langle 5|6]$,

$$P^{\langle 5|6]} A = P^{\langle 5|6]}_{[16]} A + P^{\langle 5|6]}_{s_{345}} A = c_B \frac{[65][1|2+4|3\rangle}{[15][16]\langle 32\rangle p_{234}^2} - c_A \frac{[62]}{[61]} \frac{1}{p_{345}^2}, \qquad (27)$$

where $c_A = g^2\lambda$ and $c_B = g^4$. Also $|-p\rangle = |p\rangle$ and $[-p| = -[p|$.

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- Extract the constant term with respect to $\langle 3|4]$,

$$C^{\langle 3|4]} P^{\langle 5|6]} A = c_B \frac{[65][1|2+3|4\rangle}{[15][16]\langle 42\rangle p_{234}^2} - c_A \frac{[62]}{[61]} \frac{1}{p_{345}^2}. \qquad (28)$$

- Same for $\langle 3|4]$,

$$P^{\langle 3|4]}A + C^{\langle 3|4]}P^{\langle 5|6]}A = c_B\left(-\frac{\langle 34\rangle[6|1+5|2\rangle}{\langle 24\rangle[16]\langle 32\rangle p_{234}^2} + \frac{[65][1|2+3|4\rangle}{[15][16]\langle 42\rangle p_{234}^2}\right) + c_A\left(\frac{\langle 31\rangle}{\langle 32\rangle}\frac{1}{p_{456}^2} - \frac{[62]}{[61]}\frac{1}{p_{345}^2}\right). \tag{29}$$

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- Extract the constant term with respect to $\langle 4|5]$,

$$\begin{aligned} & C^{\langle 4|5]}P^{\langle 3|4]}A + C^{\langle 4|5]\langle 3|4]}P^{\langle 5|6]}A \\ = & -c_B\frac{\langle 35\rangle[64]}{[16]\langle 32\rangle\langle 5|1+6|4]} + c_A\left(\frac{\langle 31\rangle}{\langle 32\rangle}\frac{1}{p_{456}^2} - \frac{[62]}{[61]}\frac{1}{p_{345}^2}\right). \end{aligned} \tag{30}$$

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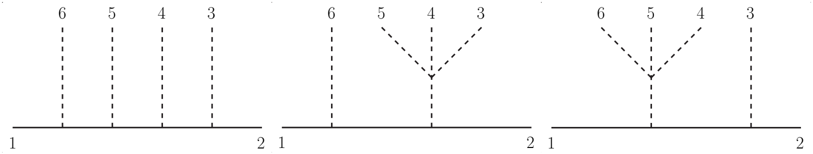
This is the correct answer, which matches the result by Feynman diagrams.



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- In this case, to extract constant terms is much easier.

Conclusion

- To judge the correct result, there are three criteria:
  (1) All spurious poles must be canceled out;
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- Thank you!

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