Energy-energy correlations: from QCD to $\mathcal{N} = 4$ SYM and back

Gregory Korchemsky
IPhT, Saclay

In collaboration with
Andrei Belitsky, Stefan Hohenegger, Emeri Sokatchev, Alexander Zhiboedov

IAS Focused Program of Scattering Amplitudes in Hong Kong
Why we like $\mathcal{N} = 4$ SYM

- Supersymmetric cousin of QCD (extended spectrum of on-shell states)
- Maximally supersymmetric, conformal four-dimensional gauge theory
- Is believed to be integrable, in the planar limit at least
- Remarkable progress in computing scattering amplitudes and correlation functions
- Use $\mathcal{N} = 4$ SYM for developing new approaches to computing physical observables in QCD

This talk:

*Compute energy-energy correlations in $\mathcal{N} = 4$ SYM at weak and at strong coupling*
$e^+e^-$ annihilation in QCD

✓ PETRA (1978-1986) and LEP (1989-2010)

✓ A virtual photon or $Z^0$—boson decay into quarks and gluons that undergo a hadronization process into hadrons

✓ Final states can be described using the class of *infrared finite* observables (event shapes): energy-energy correlations (EEC), thrust, heavy mass, . . .

✓ Can be computed in perturbative QCD, hadronisation corrections are ‘small’ at high energy
Energy-energy correlations

- Function of the angle $0 \leq \chi \leq \pi$ between detected particles
  
  $\text{EEC}(\chi) = \left\langle \frac{1}{\Delta \chi} \sum_{a,b} \frac{E_a E_b}{Q^2} \theta(\Delta \chi - |\cos \theta_{ab} - \cos \chi|) \right\rangle_{\text{events}}$

  Total energy $\sum_a E_a = Q$

- Conventional (‘amplitude’) approach
  
  $\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b} \int d\sigma_{a+b+x} \frac{E_a E_b}{Q^2} \delta(\cos \theta_{ab} - \cos \chi)$

- Weak coupling expansion in QCD
  
  $\text{EEC}(\chi) = a_s A(\chi) + a_s^2 B(\chi) + O(a_s^3)$

- Current status (1978 – today):
  - Very precise experimental data
  - Poor analytical control, $B(\chi)$ is known numerically

- Final goal: develop more efficient method to computing EEC
$e^+e^-$ annihilation in $\mathcal{N}=4$ SYM

✔ Define EEC in $\mathcal{N}=4$ SYM and evaluate it at weak/strong coupling

From QCD to $\mathcal{N}=4$ SYM: introduce an analog of the electromagnetic current

✗ (protected) half-BPS operator built from complex scalar field

$O_{20}'(x) = \text{tr} \, [\Phi \Phi]$

✗ To lowest order in the coupling, $O_{20}'(x)$ produces a pair of scalars out of the vacuum

✗ For arbitrary coupling, the state $O_{20}'(x)|0\rangle$ can be decomposed into an infinite sum over on-shell states with an arbitrary number of scalars $(s)$, gauginos $(\lambda)$ and gauge fields $(g)$

$$\int d^4x \, e^{i q \cdot x} \, O_{20}'(x)|0\rangle = |ss\rangle + |ssg\rangle + |s\lambda\lambda\rangle + \ldots$$


**EEC in $\mathcal{N} = 4$ SYM**


- **Conventional approach**

  $$\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b,X} \int \text{d}LIPS |A_{a+b+X}|^2 \frac{E_a E_b}{Q^2} \delta(\cos \chi - \cos \theta_{ab})$$

- **The amplitude of creation of the final state $|a, b, X = \text{everything}\rangle$**

- **Matrix elements** ($s_{ij} = (p_i + p_j)^2$ with $p_i^2 = 0$)

  $$|A_{ss}|^2 = \left| \langle s(p_1)s(p_2)|O_{20'}|0\rangle \right|^2 = \frac{2}{s_{12}} \left[ 1 + aF_{\text{virt}}(q^2) \right]$$

  $$|A_{ssg}|^2 = \left| \langle s(p_1)s(p_2)g(p_3)|O_{20'}|0\rangle \right|^2 = a \frac{s_{12}}{s_{13}s_{23}}$$

  $$|A_{s\lambda\lambda}|^2 = \left| \langle \lambda(p_1)\lambda(p_2)s(p_3)|O_{20'}|0\rangle \right|^2 = a \frac{2}{s_{12}}$$

- **'t Hooft coupling**

  $$a = g_{\text{YM}}^2 \frac{N}{4\pi^2}$$
The total cross section

\[ \sigma_{\text{tot}}(q) = \int \text{dLIPS}_2 |A_{ss}|^2 + \int \text{dLIPS}_3 (|A_{ssg}|^2 + |A_{s\lambda\lambda}|^2) + O(a^2) \]

\[ = \frac{N^2 - 1}{16\pi} [1 + aF_{\text{virt}}(q^2)] + a \int \text{dLIPS}_3 \frac{s_{12}^2 + 2s_{13}s_{23}}{s_{12}s_{13}s_{23}} + O(a^2) = \frac{N^2 - 1}{16\pi} + 0 \cdot a + O(a^2) \]

Energy-energy correlations

\[ \text{EEC} = \left[ \int \text{dLIPS}_2 w(p_1, p_2)|A_{ss}|^2 + \int \text{dLIPS}_3 w(p_1, p_2, p_3)(|A_{ssg}|^2 + |A_{s\lambda\lambda}|^2) + O(a^2) \right] / \sigma_{\text{tot}} \]

Weight factors for EEC

\[ w(p_1, p_2, \ldots) = \sum_{a,b} \frac{E_a E_b}{q^2} \delta(\cos \theta_{ab} - \cos \chi) \]

One-loop calculation (unprotected quantity) \[\text{[Zhiboedov],[Engelund,Roiban]}\]

\[ \text{EEC}_{\mathcal{N}=4} = \frac{a}{4z^2(1 - z)} \ln \frac{1}{1 - z} + O(a^2) \]

IR finite, positive definite function of \( z = (1 - \cos \chi)/2 , \quad 0 < z < 1 \)

Two-loop correction is hard to compute (\( \sim 10^2 \) diagrams)
**EEC from amplitudes II**

✔ Conventional approach

\[
\text{EEC}(\chi) = \frac{1}{\sigma_{\text{tot}}} \sum_{a,b,X} \int \text{dLIPS} \ |A_{a+b+X}|^2 \frac{E_a E_b}{Q^2} \delta(\cos \chi - \cos \theta_{ab})
\]

The amplitude of creation of the final state \(|a, b, X = \text{everything}\rangle\)

\[
A_{a+b+X} = \int d^4 x \ e^{i q x} \langle a, b, X | O_{20'}(x) | 0 \rangle
\]

✔ Main disadvantages:

✗ presence of infrared divergences in transition amplitudes \(A_{a+b+X}\)

✗ integration over the Lorentz invariant phase space of the final states \(d\text{LIPS}\)

✗ necessity for summation over all final states \(\sum X\)

✗ no analytical results beyond one loop

✔ New approach: EEC can be computed from *correlation functions of energy flow operators*
**EEC from correlation functions**

- **Total cross section from the optical theorem**

\[
\sigma_{\text{tot}}(q) = \sum_X (2\pi)^4 \delta^{(4)}(q - p_X) |A_{O_{20}^t \rightarrow X}|^2
\]

\[
= \int d^4x \ e^{iqx} \sum_X \langle 0| O^\dagger(0)|X \rangle e^{-ipX} \langle X| O(0)|0 \rangle
\]

\[
= \int d^4x \ e^{iqx} \langle 0| O^\dagger(x) O(0)|0 \rangle = \frac{1}{16\pi} (N^2 - 1) \theta(q^0) \theta(q^2)
\]

*Wightman correlation function, protected for 1/2-BPS operators*

- **Generalization to EEC**

\[
\text{EEC} \sim \sum_X \langle 0| O^\dagger(x)|X \rangle w(X) \langle X| O(0)|0 \rangle = \langle 0| O^\dagger(x) \mathcal{E}(\vec{n}_1) \mathcal{E}(\vec{n}_2) O(0)|0 \rangle
\]

*Energy flow operator*

\[
\mathcal{E}(\vec{n})|X\rangle = \sum_a E_a \delta^{(2)}(\Omega_{\vec{p}_a} - \Omega_{\vec{n}})|X\rangle
\]

- **Relation to the energy-momentum tensor in $\mathcal{N} = 4$ SYM**

[Sveshnikov,Tkachov],[GK,Oderda,Sterman]

\[
\mathcal{E}(\vec{n}) = \int_0^\infty dt \ \lim_{r \rightarrow \infty} r^2 \vec{n}^i T_{0i}(t, r\vec{n})
\]
EEC from correlation functions II

✔ Energy flow correlations \([\text{GK, Sterman}, \text{Belitsky, GK, Sterman}, \text{Hofman, Maldacena}]\)

\[
\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_q = \sigma_{tot}^{-1} \int d^4x \ e^{iqx} \langle 0| \hat{O}\dagger(x) \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) O(0)|0 \rangle
\]

*Energy flow in the direction of* \(\vec{n}_1\) *and* \(\vec{n}_2\)

✔ Average over the orientations \(\vec{n}_1\) and \(\vec{n}_2\) with the relative angle \(\chi\) kept fixed

\[
\text{EEC} = \int d\Omega_1 d\Omega_2 \ \delta(\vec{n}_1 \cdot \vec{n}_2 - \cos \chi) \langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2) \rangle_q / q^2
\]

✔ Multi-fold integral of *Wightman* 4pt function

\[
\text{EEC} \sim \int d^4x \ e^{iqx} \int_0^\infty dt_1 dt_2 \ \lim_{r_i \to \infty} r_1^2 r_2^2 \langle 0| \hat{O}\dagger(x) \ T_0\vec{n}_1(x_1)T_0\vec{n}_2(x_2) O(0)|0 \rangle \bigg|_{x_i = (t, r\vec{n}_i)}
\]

✗ Compute corr.function \(\langle \hat{O}\dagger(x) T(x_1)T(x_2) O(0) \rangle\) in Euclid

✗ Continue to Minkowski with Wightman prescription

✗ Take detector limit + perform Fourier
Correlation functions in $\mathcal{N} = 4$ SYM

- Quantum corrections to various correlation functions are determined by the same scalar function

$$
\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle_E = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} \Phi(u, v; a)
$$

$$
\langle O(x_1)T(x_2)T(x_3)O(x_4)\rangle_E = \frac{1}{(x_{12}^2 x_{23}^2 x_{34}^2)^2} P(\partial_u, \partial_v) \Phi(u, v; a)
$$

Conformal ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

- Universal function in $\mathcal{N} = 4$ SYM at weak coupling

$$
\Phi(u, v) = a \Phi^{(1)}(u, v) + a^2 \left( \frac{1}{2} (1 + u + v) \left[ \Phi^{(1)}(u, v) \right]^2 
+ 2 \left[ \Phi^{(2)}(u, v) + \frac{1}{u} \Phi^{(2)}(v/u, 1/u) + \frac{1}{v} \Phi^{(2)}(1/v, u/v) \right] \right) + O(a^3)
$$

$\Phi^{(1)}(u, v)$ ‘box’ integral, $\Phi^{(2)}(u, v)$ ‘double’ box integral

- $\mathcal{N} = 4$ superconf. symmetry allows us to determine $\Phi_{\text{weak}}(u, v)$ to six loops

- AdS/CFT correspondences predicts $\Phi(u, v)$ at strong coupling
From Euclid to Minkowski

✔ Brute force method: compute anew using Schwinger-Keldysh technique (too hard)

✔ Better method: analytically continue correlation functions from Euclid to Minkowski+Wightman

✔ Warm-up example: free scalar propagator $D_{\text{Euclid}}(x) = \langle \phi(x)\phi(0) \rangle \sim 1/x^2$

$$\langle 0|\phi(x)\phi(0)|0 \rangle = \sum_{n} \langle 0|\phi(x)|n \rangle \langle n|\phi(0)|0 \rangle$$

$$= \sum_{E_n > 0} e^{-iE_n(x^0 - i0) + i\vec{p}\cdot\vec{x}} \langle 0|\phi(0)|n \rangle \langle n|\phi(0)|0 \rangle \sim \frac{1}{(x^0 - i0)^2 - \vec{x}^2}$$

✔ How to get Wightman correlation functions ('magic' recipe)

✗ Go to Mellin space:

$$\Phi_{\text{Euclid}} = \int_{-\delta - i\infty}^{-\delta + i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) u^{j_1} v^{j_2} , \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$$

✗ Substitute $x_{ij}^2 \rightarrow x_{ij,+}^2 = x_{ij}^2 - i0 \cdot x_{ij}^0$

$$\Phi_{\text{Wightman}} = \int_{-\delta - i\infty}^{-\delta + i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M(j_1, j_2; a) \left( \frac{x_{12}^2 + x_{34}^2}{x_{13}^2 + x_{24}^2} \right)^{j_1} \left( \frac{x_{23}^2 + x_{41}^2}{x_{13}^2 + x_{24}^2} \right)^{j_2}$$

✔ $M(j_1, j_2; a)$ is known both at weak and strong coupling in planar $\mathcal{N} = 4$ SYM
All-loop prediction for EEC

Master formula

\[ \text{EEC}(\chi) = \frac{1}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1dj_2}{(2\pi i)^2} M(j_1, j_2; a) K(j_1, j_2) \left( \frac{1 - z}{z} \right)^{j_1 + j_2} \]

The dependence on the angle \( \chi \) enters through

\[ z = \frac{(1 - \cos \chi)}{2}, \quad 0 < z < 1 \]

Detector function is independent on the coupling

\[ K(j_1, j_2) = \frac{2 \Gamma(1 - j_1 - j_2)}{\Gamma(j_1 + j_2)[\Gamma(1 - j_1)\Gamma(1 - j_2)]^2} \]

The dependence on the coupling constant resides in the Mellin amplitude

\[ \Phi(u, v; a) = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1dj_2}{(2\pi i)^2} M(j_1, j_2; a) u^{j_1}v^{j_2} \]

\[ M(j_1, j_2; a) = aM^{(1)}(j_1, j_2) + a^2M^{(2)}(j_1, j_2) + \ldots \]

are known
Warm up exercise

✔ Master formula at one loop

\[
\text{EEC}^{(1\text{-loop})} = \frac{a}{4z^2(1-z)} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1 dj_2}{(2\pi i)^2} M^{(1)}(j_1, j_2; a) K(j_1, j_2) \left( \frac{1-z}{z} \right)^{j_1+j_2}
\]

Mellin amplitude

\[
M^{(1)}(j_1, j_2) = -\frac{1}{4} \left[ \Gamma(-j_1) \Gamma(- j_2) \Gamma(1 + j_1 + j_2) \right]^2
\]

\[
K(j_1, j_2) = \frac{2 \Gamma(1 - j_1 - j_2)}{\Gamma(j_1 + j_2) [\Gamma(1 - j_1) \Gamma(1 - j_2)]^2}
\]

✔ Change integration variable \( j_1 + j_2 \to j_1 \)

\[
\text{EEC}^{(1\text{-loop})} = -\frac{a}{4z^2(1-z)} \int \frac{dj_1 dj_2}{(2\pi i)^2} \frac{j_1^2}{2(j_1 - j_2)^2 j_2^2} \frac{\pi}{\sin(\pi j_1)} \left( \frac{1-z}{z} \right)^{j_1}
\]

\[
= \frac{a}{4z^2(1-z)} \int \frac{dj_1}{2\pi i} \frac{\pi}{j_1 \sin(\pi j_1)} \left( \frac{1-z}{z} \right)^{j_1}
\]

\[
= \frac{a}{4z^2(1-z)} \sum_{k=-\infty}^{-1} \frac{(-1)^k}{k} \left( \frac{1-z}{z} \right)^k
\]

\[
= \frac{a}{4z^2(1-z)} \ln \frac{1}{1-z}
\]
**EEC at two loops**

Final result for EEC

\[
\text{EEC}_{\mathcal{N}=4} = \frac{1}{4z^2(1-z)} \left\{ aF_1(z) + a^2 \left[ (1 - z)F_2(z) + \frac{1}{4} F_3(z) \right] + O(a^3) \right\}, \quad z = \frac{1}{2}(1 - \cos \chi)
\]

\(F_w(z)\) are linear combinations of functions of homogenous weight \(w = 1, 2, 3\)

\[
F_1(z) = -\ln(1 - z)
\]

\[
F_2(z) = 4\sqrt{z} \left[ \text{Li}_2 \left( -\sqrt{z} \right) - \text{Li}_2 \left( \sqrt{z} \right) + \frac{1}{2} \ln z \ln \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]
+ (1 + z) \left[ 2\text{Li}_2(z) + \ln^2(1 - z) \right] + 2 \ln(1 - z) \ln \left( \frac{z}{1 - z} \right) + z \frac{\pi^2}{3},
\]

\[
F_3(z) = (1 - z)(1 + 2z) \left[ \ln^2 \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \ln \left( \frac{1 - z}{z} \right) - 8\text{Li}_3 \left( \frac{\sqrt{z}}{\sqrt{z} - 1} \right) - 8\text{Li}_3 \left( \frac{\sqrt{z}}{\sqrt{z} + 1} \right) \right]
- 4(z - 4)\text{Li}_3(z) + 6(3 + 3z - 4z^2)\text{Li}_3 \left( \frac{z}{z - 1} \right) - 2z(1 + 4z)\zeta_3 + 2[(3 - 4z)z \ln z
+ 2(2z^2 - z - 2) \ln(1 - z)] \text{Li}_2(z) + \frac{1}{3} \ln^2 (1 - z) \left[ 4(3z^2 - 2z - 1) \ln(1 - z)
+ 3(3 - 4z)z \ln z \right] + \frac{\pi^2}{3} \left[ 2z^2 \ln z - (2z^2 + z - 2) \ln(1 - z) \right]
\]
From weak to strong coupling

- At weak coupling $\text{EEC}_{\mathcal{N}=4}$ has a shape which is remarkably similar to the one in QCD.

- Going from one to two loops, EEC flattens.

- This agrees with strong coupling prediction for EEC in planar $\mathcal{N} = 4$ SYM [Hofman,Maldacena]

\[ \text{EEC}_{\mathcal{N}=4} \xrightarrow{\alpha \to \infty} \frac{1}{2} \left[ 1 + a^{-1} (1 - 6z(1 - z)) + O(a^{-3/2}) \right] \]

No jets at strong coupling.
End-point asymptotics I

Small angle correlations $\chi \to 0$ (or $z \sim \chi^2 \to 0$): calorimeters measure nearly collinear particles

$$EEC \underset{z \to 0}{\sim} \frac{a}{4z} \left[ 1 + a \left( \ln z - \frac{1}{2} \zeta_3 + \zeta_2 - 3 \right) \right]$$

- Corrections are enhanced by $\ln z$, no homogenous transcendentality
- Resummation of leading log's $a(a \ln z)^k$ using the "jet calculus" [Konishi,Ukawa,Veneziano]

$$EEC \underset{z \to 0}{\sim} \frac{a}{4z} \int_0^1 dx x^2 D(x, Q^2/S_{ab})$$

$$= \frac{a}{4z} (Q^2/S_{ab})^{-\gamma_T(3)} = \frac{a}{4} z^{-1+\gamma_T(3)}$$

$$\gamma_T(3) = a + O(a^2) - \text{twist-2 time-like anomalous dimension of spin } S = 3$$

$$D(x, Q^2/S_{ab}) \text{ probability to fragment into a pair of partons with } S_{ab} = 2 E_a E_b (1 - \cos \chi) \sim Q^2 z$$

- Resummation weakens singularity of EEC for $\chi \to 0$, jets at weak coupling

$$\int_0^{\chi_0} d\cos \chi \ EEC \sim \frac{a}{\gamma_T(3)} \sim 1, \quad (\chi_0 \ll 1)$$
End-point asymptotics II

EEC in the back-to-back kinematics $\chi \to \pi$ (or $y \equiv 1 - z \sim (\pi - \chi)^2 \to 0$)

\[ \text{EEC} \left( \frac{1}{4y} \right) \left\{ a \ln(1/y) - \frac{a^2}{2} \left[ \ln^3(1/y) + \frac{\pi^2}{2} \ln(1/y) \right] \right\} \]

✔ Large (Sudakov) corrections $a^k (\ln y)^n$ come from the emission of soft and collinear particles

✔ All order resummation

\[ \text{EEC} \sim \frac{1}{8y} H(a) \int_0^\infty db \, b J_0(b) S(b^2/y; a) \]

$J_0(b)$ Bessel function; $S(b^2/y; a)$ the Sudakov form factor (with $b_0 = 2 \, e^{-\gamma_E}$)

\[ S = \exp \left[ -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln^2 \left( \frac{b^2}{yb_0^2} \right) - \Gamma(a) \ln \left( \frac{b^2}{yb_0^2} \right) \right] \]

Dependence on the coupling constant is encoded in three functions

\[ \Gamma_{\text{cusp}}(a) = a - \frac{1}{2} \zeta_2 a^2, \quad \Gamma(a) = -\frac{3}{2} \zeta_3 a^2, \quad H(a) = 1 - \zeta_2 a \]

✔ Perturbative corrections to $\text{EEC}(z \to 1)$ have homogeneous transcendentality

\[ \left[ \text{EEC}_{\text{QCD}}(z \to 1) \right] \text{maximal transcendentality} = \text{EEC}_{N=4}(z \to 1) \]
Conclusions and open questions

✔ Energy correlations are good/nontrivial physical observables in $\mathcal{N} = 4$ SYM

✔ Relation to energy-energy correlations in QCD (most complicated part)?

✔ All symmetries of $\mathcal{N} = 4$ SYM are preserved, what is the manifestation of integrability?

✔ Interpolation between weak and strong coupling?

✔ Other proposals for ‘good’ observables?