Logarithmic Singularities of $\mathcal{N} = 4$ Super-Yang-Mills Amplitudes

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Outline

Introduction

Arkani-Hamed, Bourjaily, Cachazo and Trnka [arXiv:1410.0354]

> *To all orders of perturbation theory, scattering amplitudes in $\mathcal{N} = 4$ SYM beyond the planar limit have only logarithmic singularities, without any poles at infinity.*

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- link to basis of integrals of maximal uniform transcendentality ✗

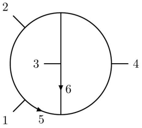
  e.g. Henn, Smirnov^2 [arXiv:1306.2799]
- relation to singularity structure of gravity amplitudes via BCJ ✗

  Bern, Carrasco, Johansson [arXiv:0805.3993]

General Strategy

non-planar amplitude $\rightarrow \nexists$ unique integrand $\Rightarrow$ expand amplitude in integral basis

1. define set $\mathcal{S}$ of parent diagrams in cubic graph representation (no triangle or bubble subdiagrams)



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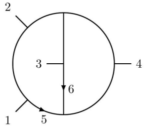

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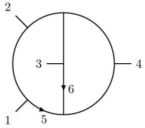

$\Rightarrow$ Basis of integrals[1]:

$$\int d\mathcal{I}^{(x)} \equiv \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \frac{st A_4^{\text{tree}} \sum_k a_k^{(x)} N_k^{(x)}(\ell_i, p_j)}{\prod_{\alpha(x)} P_{\alpha(x)}(\ell_i, p_j)}$$

[1] $P_{\alpha(x)}(\ell_i, p_j)$ are Feynman propagators

3. expand amplitude in integral basis

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{\sigma(m)} \sum_{x\in\mathcal{S}} \frac{c^{(x)}}{S^{(x)}} \int d\mathcal{I}^{(x)}(\ell_1, ..., \ell_L, p_1, ..., p_m)$$

4. Use unitarity cuts or leading singularity methods to determine coefficients $a_k^{(x)}$ e.g. Bern, Dixon, Dunbar, Kosower [arXiv:9409265] & Cachazo [arXiv:0803.1988]

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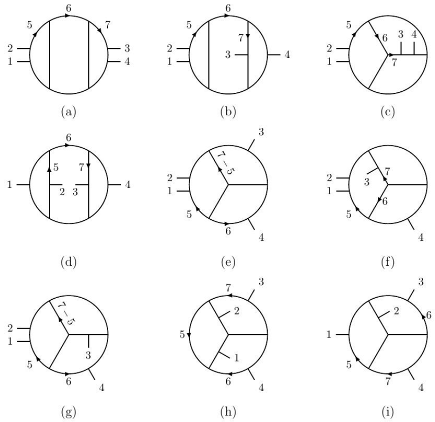
5 Use method of maximal cuts to confirm that amplitude is correct and complete e.g. Bern, Carrasco, Johansson, Kosower [arXiv:0705.1864]

3-loop basis of Integrals

1. Parent diagrams for 3-loop amplitude in $\mathcal{N} = 4$ SYM ✓

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban [arXiv:07022112, 0808.4112]



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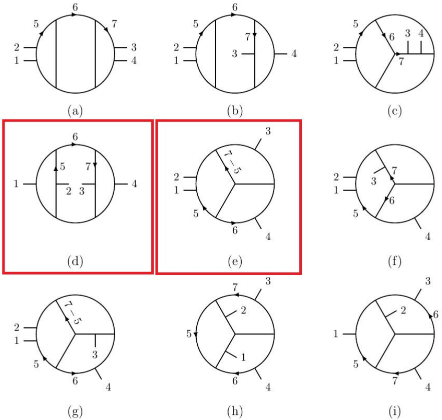
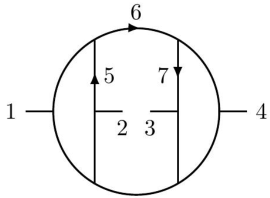


Example 1 - Diagram d)

2 basis of numerators $N_k^{(d)}$

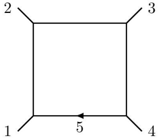
- diagram has only Feynman propagators $\rightarrow$ double poles?
- go deeper in the cut structure of the integral



Useful cut technology

recurring cut situations

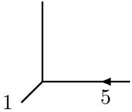
1. box cut



$$\ell_5^* = \cdots\,, \quad J_5 = \cdots$$

2. collinear cut

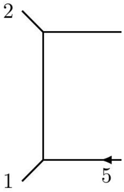
   completely massless corner



$$\ell_5^* = \alpha k_1$$

3. soft-collinear cut

   completely massless rung



$$\ell_5^* = k_1, \quad J_5 = (k_1+k_2)^2$$

For box Jacobians and cut solutions, see e.g. Britto, Cachazo, Feng [arXiv:0412103]

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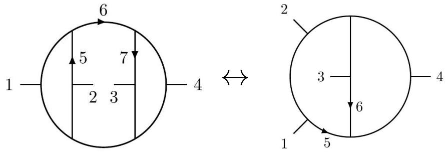
2 basis of numerators $N_k^{(d)}$

- power counting for numerator
  $N_k^{(d)} = \mathcal{O}(\ell_6^4) = (\ell_6^2 + Q_1 \cdot \ell_6 + c_1)(\ell_6^2 + Q_2 \cdot \ell_6 + c_2)$
- resemblance with 2-loop nonplanar diagram

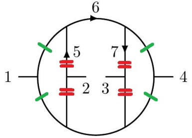
$$N_{\text{2-loop}}^{(\mathrm{NP})} = (\ell_5 - k_3)^2 + (\ell_5 - k_4)^2$$

$\Rightarrow$ Ansatz for 3-loop diagram

$$\widetilde{N}^{(\mathrm{d})} = \left[(\ell_6 + k_1)^2 + (\ell_6 + k_2)^2\right] \cdot \left[(\ell_6 - k_3)^2 + (\ell_6 - k_4)^2\right]$$



- Ansatz free of double poles?
- $\widetilde{N}^{(\mathrm{d})} = \left[(\ell_6 + k_1)^2 + (\ell_6 + k_2)^2\right] \cdot \left[(\ell_6 - k_3)^2 + (\ell_6 - k_4)^2\right]$



(d)

- Collinear cuts $\Rightarrow \ell_5 = \alpha k_2,\ \ell_7 = -\beta k_3$
- quadruple cut $\Rightarrow$ Jacobian $J_6 = su(\alpha - \beta)^2$,

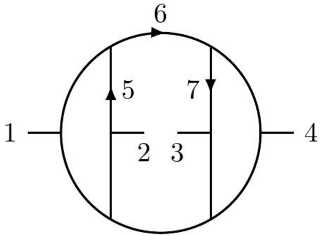
$$\ell_6^* = \alpha\lambda_4\widetilde{\lambda}_2\frac{\langle 12\rangle}{\langle 14\rangle} - \beta\lambda_1\widetilde{\lambda}_3\frac{\langle 34\rangle}{\langle 14\rangle}$$

double pole! ✗

$$\text{Res}\left[\mathcal{I}^{(\mathrm{d})}\right]_{\text{cut}} = -\frac{(\alpha(1+\beta)+\beta(1+\alpha))^2}{s^3 u\alpha\beta(1+\alpha)(1+\beta)(\alpha-\beta)^2}$$

2 basis of numerators $N_k^{(d)}$

- Can we cancel the double pole?
- Add contact term
  $\ell_6^2(\ell_6+k_1+k_2)^2 \to s^2\alpha\beta(\alpha+1)(\beta+1)$

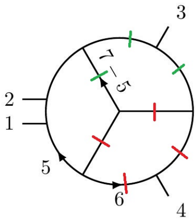


final numerator - no double pole ✓

$$N^{(\mathrm{d})} = \left[(\ell_6+k_1)^2 + (\ell_6+k_2)^2\right] \cdot \left[(\ell_6-k_3)^2 + (\ell_6-k_4)^2\right] - 4\ell_6^2(\ell_6+k_1+k_2)$$

Example 2 - Diagram e)

- planar diagram → DCI → numerator
- make no use of DCI, but derive numerator from log. sing. constraints
- power counting for numerator
  $N_k^{(e)} = \mathcal{O}(\ell_5^2) = (c_1 s + c_2 t)(\ell_5^2 + d_1 Q \cdot \ell_5 + d_2 s + d_3 t)$



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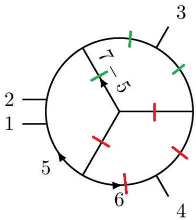

  - extract double pole constraints $\rightarrow$ quadruple cut on ℓ_6-box subdiagram
    $J_6 = (\ell_5 + k_4)^2 \ell_7^2 - \ell_5^2 (\ell_7 + k_4)^2$
  - cut $(\ell_7 + k_4)^2 \Rightarrow$ factorizes $J_6 \rightarrow (\ell_5 + k_4)^2 \ell_7^2$
    generates new propagator ℓ_7^2

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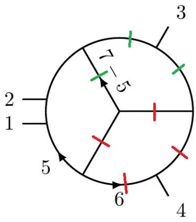

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    generates new propagator ℓ_7^2
  - also set $\ell_7^2 = (\ell_7+k_3+k_4)^2 = (\ell_7-\ell_5)^2 = 0 \Rightarrow J_7 = s(\ell_5+k_4)^2$
  - Res $\left[\mathcal{I}^{(e)}\right]_{\text{cut}} = \frac{N^{(e)}}{s\ell_5^2(\ell_5-k_1)^2(\ell_5-k_1-k_2)^2(\ell_5+k_4)^4}$

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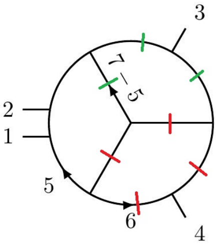

final numerator - no double pole ✓

$$N^{(e)} = (c_1 s + c_2 t)(\ell_5 + k_4)^2$$

- $N^{(e)}$ agrees with the DCI numerator! (up to factor independent of loop momenta)

Why back to planar sector?

- dual formulation of scattering amplitudes in terms of on-shell diagrams and cells in the positive Grassmannian
  $\Rightarrow$ logarithmic singularities and no poles at infinity manifest
- interplay with local diagrammatic expansion? $\rightarrow$ use same strategy as in nonplanar sector
- rather than an input, can we see DCI emerge?

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Drummond, Korchemsky, Sokatchev [arXiv:0707.0243]

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Logarithmic singularities and absence of poles at infinity imply dual conformal symmetry of local integrals in the planar sector.

To cancel multiple poles $\rightarrow$ link various DCI-integrals together

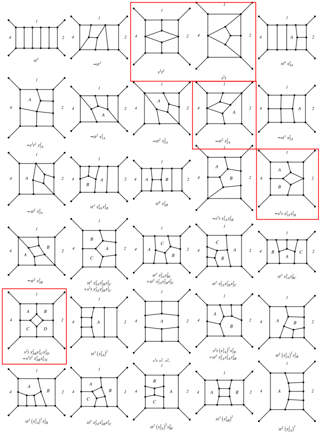
5-loop planar integrals

Integrals assembling the 5-loop amplitude in planar $\mathcal{N} = 4$ SYM

(nonzero coefficients)

Bern, Carrasco, Johansson, Kosower [arXiv:0705.1864],

Bourjaily, DiRe, Shaikh, Spradlin, Volovich [arXiv:1112.6432]

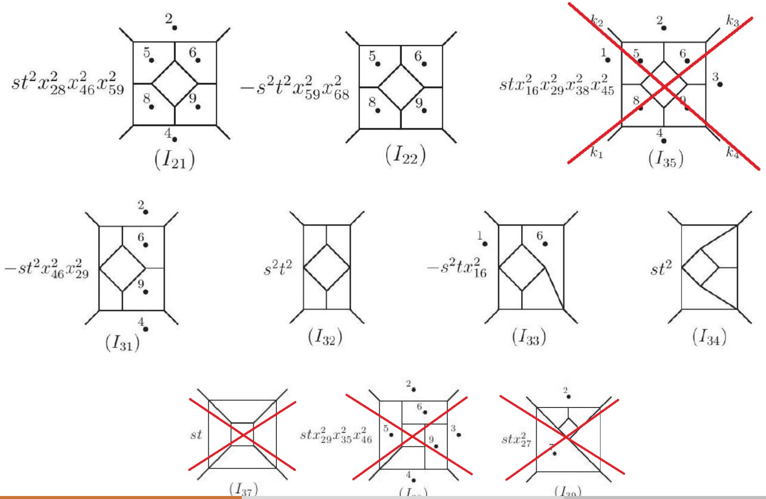


Example 1 - Window Diagram at 5-loops

Bern, Carrasco, Johansson, Kosower [arXiv:0705.1864]

Bourjaily, DiRe, Shaikh, Spradlin, Volovich [arXiv:1112.6432]

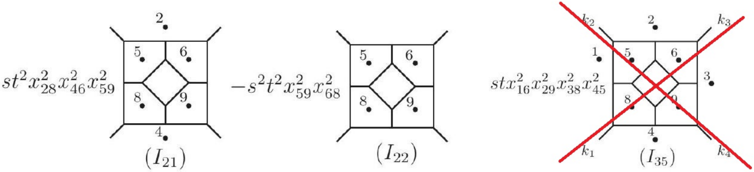
✗ = coefficient zero



Example 1 - Coefficient 0 Window Diagram

- All coefficient zero integrals explained up to 5-loops

  Drummond, Korchemsky, Sokatchev [arXiv:0707.0243]

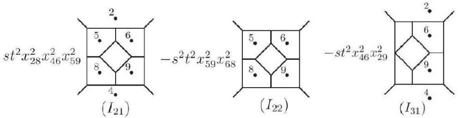


- limit where x_6, x_7, x_8, x_9 approach internal point x_5
  $\rho^2 = x_{56}^2 + x_{57}^2 + x_{58}^2 + x_{59}^2 \to 0$

$$\int \frac{d^4x_6 d^4x_7 d^4x_8 d^4x_9 N^{(j)}(x_5, x_6, x_8, x_9)}{x_{56}^2 x_{57}^2 x_{58}^2 x_{67}^2 x_{69}^2 x_{78}^2 x_{79}^2 x_{89}^2} \sim \int \frac{\rho^{15} N^{(j)} d\rho}{\rho^{16}}$$

- no details, but coefficient zero integrals have double poles!
- logarithmic singularities does more $\rightarrow$ links DCI integrals together, explains coefficient zero integrals at higher loop beyond [arXiv:0707.0243]

- Example: beyond maximal cut of I_{31}
  (cutting Jacobians) $\rightarrow$ localize x_5, x_6, x_8 and x_9



$$\mathrm{Res}\left[\mathcal{I}_{21}+\mathcal{I}_{22}+\mathcal{I}_{31}\right]_{\mathrm{cut}} \sim \frac{\mathrm{d}^4\mathrm{x}_7}{(\mathrm{x}_{74}^2)^2\mathrm{x}_{73}^2\mathrm{x}_{72}^2}\left(\mathrm{N}_{\mathrm{cut}}^{(21)}+0+\mathrm{N}_{\mathrm{cut}}^{(31)}\right)$$

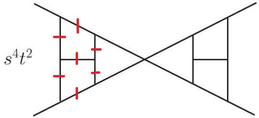
- Need to combine integrals to cancel double poles!

$$\mathcal{I}^A=\mathcal{I}_{21}+\mathcal{I}_{31}+\mathcal{I}_{34}, \qquad \mathcal{I}^B=\mathcal{I}_{22}+\mathcal{I}_{32}, \qquad \mathcal{I}^C=\mathcal{I}_{33}$$

Example 2 - Bowtie Diagram at 6-loops

Bern, Carrasco, Johansson, Kosower [arXiv:0705.1864]

Bourjaily, DiRe, Shaikh, Spradlin, Volovich [arXiv:1112.6432]



- 6-loop planar coefficient 0 integral
- so far unexplained by heuristic-rules

  Drummond, Korchemsky, Sokatchev [arXiv:0707.0243]
- cut double-box completely (including Jacobian) $\rightarrow$ double pole
- no numerator allowed to cancel

Conclusion

- logarithmic singularities and no poles at infinity of the full 3-loop amplitude in $\mathcal{N} = 4$ SYM
- relation of logarithmic singularities and dual conformal invariance in the planar amplitude up to 6-loops
- explain the absence of certain integrals in the diagrammatic expansion of high loop planar $\mathcal{N} = 4$ SYM
- unify known heuristic rules for dual conformal diagrams in a comprehensive framework

Outlook/Speculation

- Can we find a reformulation of nonplanar scattering amplitudes in $\mathcal{N} = 4$ SYM that make the properties of logarithmic singularities no poles at infinity manifest? nonplanar on-shell diagrams?
- Is there a generalization of dual conformal invariance to the nonplanar amplitude?