Tutorial of on-shell recursion relation

Bo Feng

The IAS Focused Program on Scattering Amplitudes in Hong Kong, 17-21 November 2014

Contents

Contents

Contents

Contents

A few main references:

- (I) Z. Bern, L. J. Dixon and D. A. Kosower, "On-Shell Methods in Perturbative QCD," Annals Phys. **322**, 1587 (2007) [arXiv:0704.2798 [hep-ph]].
- (II) B. Feng and M. Luo, "An Introduction to On-shell Recursion Relations," Front. Phys. **7**, 533 (2012) [arXiv:1111.5759 [hep-th]].
- (III) H. Elvang and Y. t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

Spinor Notations

- $k_{\dot{a}a} \equiv k_\mu \sigma^\mu$, $k_\mu \cdot k^\mu = \det(k_{\dot{a}a})$. Massless condition leads to

$$k_{\dot{a}a} = \widetilde{\lambda}_{\dot{a}} \lambda_a$$

where we have spinor λ_a and antispinor $\widetilde{\lambda}_{\dot{a}}$. The factorization property is the key of much simple expression of amplitudes when using spinor notation.

- Spinor indices can be raised or lowered as

$$\lambda^a = \epsilon^{ab} \lambda_b, \quad \lambda_a = \epsilon_{ab} \lambda^b , \qquad (1)$$

Thus we can also define Lorentz invariant products

$$\langle i|j \rangle \equiv \lambda_i^a \lambda_{ja}, \quad [i|j] \equiv \widetilde{\lambda}_{i\dot{a}} \widetilde{\lambda}_j^{\dot{a}} \qquad (2)$$

- To map to familiar notations, first notice that

$$\begin{aligned} u_{\pm}(k) &= \frac{1 \pm \gamma_5}{2} u(k), \quad v_{\mp}(k) = \frac{1 \pm \gamma_5}{2} u(k), && (3) \\ \overline{u_{\pm}(k)} &= \overline{u(k)} \frac{1 \mp \gamma_5}{2}, \quad \overline{v_{\mp}(k)} = \overline{v(k)} \frac{1 \mp \gamma_5}{2} && (4) \end{aligned}$$

Thus we have

$$\begin{aligned} |i\rangle &\equiv \left|k_i^+\right\rangle = u_+(k_i) = v_-(k_i), \\ |i] &\equiv \left|k_i^-\right\rangle = u_-(k_i) = v_+(k_i), \\ \langle i| &\equiv \left\langle k_i^-\right| = \overline{u_-}(k_i) = \overline{v_+}(k_i), \\ [i| &\equiv \left\langle k_i^+\right| = \overline{u_+}(k_i) = \overline{v_-}(k_i) && (5) \end{aligned}$$

- Using above notation, we have following translations

$$\langle i|j\rangle = \overline{u_-}(k_i)u_+(k_j), \quad \langle i|P|j] = \overline{u_-}(k_i)\ \not{P}u_-(k_j)$$
$$\overline{u_+(k_i)}\not{k}_j u_+(k_l) \equiv \left[i|k_j|l\right\rangle, \quad \overline{u_+(k_i)}\not{k}_j\not{k}_m u_-(k_l) \equiv \left[i|k_jk_m|l\right]$$

- One most important fact is the polarization vector can be written as

$$\epsilon^+_\nu(k|\mu) = \frac{+\langle\mu|\gamma_\nu|k]}{\sqrt{2}\,\langle\mu|k\rangle}, \quad \epsilon^-_\nu(k|\mu) = \frac{-[\mu|\gamma_\nu|k\rangle}{\sqrt{2}\,[\mu|k]},$$

The key is to use right variables: Changing variables from (k_μ, ϵ_μ) to $(\lambda, \widetilde{\lambda})$, especially for the on-shell massless particles. [Xu, Zhang, Chang, 1987]

- momentum $k \to \lambda, \widetilde{\lambda}$.
- For scalar, wave function 1.
- For massless fermions, wave functions $\lambda, \widetilde{\lambda}$
- For vector, wave functions $\epsilon^{\pm}_{\nu}(k|\mu)$.

Color ordering

- Generators of $SU(N_c)$ in the fundamental representation can be taken as

$$\mathrm{Tr}(T^a T^b) = \delta_{ab}, \quad f^{abc} = \frac{-i}{\sqrt{2}} \mathrm{Tr}(T^a [T^b, T^c]), \tag{6}$$

thus

$$\sum_{a=1}^{N_c^2-1} (T^a)_{i_1}^{\bar{j}_1} (T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c} \delta_{i_1}^{\bar{j}_1} \delta_{i_2}^{\bar{j}_2} \tag{7}$$

or equivalent to be

$$\sum_a \mathrm{Tr}(XT^a)\mathrm{Tr}(T^a Y) = \mathrm{Tr}(XY) - \frac{1}{N_c}\mathrm{Tr}(X)\mathrm{Tr}(Y)$$

$$\sum_a \mathrm{Tr}(XT^a Y T^a) = \mathrm{Tr}(X)\mathrm{Tr}(Y) - \frac{1}{N_c}\mathrm{Tr}(XY) . \tag{8}$$

- Color ordering: Thus we write whole amplitudes into gauge invariant subset (the color-ordered amplitudes)

$$M^{tree}(1,2,...,n) = \sum_{permutation} \mathrm{Tr}(T_{a_1}...T_{a_n})A_n^{tree}(a_1,a_2,...,a_n)$$

- Color ordering separate the group information from the dynamical information.
- Naively there are $(n-1)!$ different dynamical basis, but there are some relations among them to reduce to independent basis $(n-3)!$.

Four relations for ordered gluon amplitudes:

- Color-order reversed relation:

$$A(n,\{\beta_1,...,\beta_{n-2}\},1)=(-)^n A(1,\beta_{n-2},\beta_{n-1},...,\beta_1,n)$$

- The $U(1)$-decoupling relation is given by

$$\sum_{\sigma\in cyclic} A_n(1,\sigma(2,3,...,n))=0$$

- KK-relation: [Kleiss, Kujif, 1989]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n) \ .$$

  where sum is over partial ordering.
- Example

$$A(1,\{2\},5,\{3,4\} = A(1,2,4,3,5)$$
$$+A((1,4,2,3,5) + A(1,4,3,2,5)$$

BCJ-relation: [Bern, Carraso, Johansson, 2008]

$$A_n(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\sigma_i \in POP} A_n(1,2,3,\sigma_i)\mathcal{F},$$

$$\alpha = \{4,5,...,m\}$$

$$\beta = \{m+1, m+2, ..., n\}$$

- Beautiful proof from string theory

  [Bjerrum-Bohr, Damgaard, Vanhove, 2009]

  [Stieberger, 2009]

- Pure field theory proof

  [Feng, Huang, Jia, 2010]

  [Chen, Du, Feng, 2011]

Part II: Derivation of on-shell (BCFW) recursion relation

Structure for Tree-level amplitudes: $\Longleftarrow$ It will be used late:

- Only singularity is poles. From Feynman diagrams, it appears when propagators are on-shell.
- Factorization property: When one propagator goes to on-shell, i.e., $P^2 - m^2 \to 0$, we have

$$A^{tree}(1,..,n) \rightarrow \sum_{\lambda} A_{m+1}(1,..,m,P^{\lambda})\frac{1}{P_{1m}^2 - m^2} A_{n-m+1}(-P^{-\lambda}, m+1,...,n)$$

In fact, this point gives the residue at the pole.

BCFW deformation

- One basic assumption: Tree-level amplitude $\mathcal{M}$ can be considered as a rational function of complex momenta.
- BCFW deformation: Let us consider following deformation. Picking two external momenta p_1, p_2 and auxiliary momentum q, we do following deformation:

$$p_1(z) = p_1 + zq, \qquad p_2(z) = p_2 - zq$$

and impose following conditions:

$$q^2 = q \cdot p_1 = q \cdot p_2 = 0$$

[Britto, Cachazo, Feng , 2004] [Britto, Cachazo, Feng , Witten, 2004]

BCFW recursion relation

Two good points of BCFW deformation:

- It keeps the momentum conservation conditions: $p_1 + p_2 = p_1(z) + p_2(z)$
- It keeps on-shell conditions $p_1^2 = p_1(z)^2$, $p_2^2 = p_2(z)^2$;
- Amplitude becomes the meromorphic function of single complex variable z. $(P + zq)^2 = P^2 + z(2P \cdot q)$. $\Longleftarrow$ Much easy to study.

BCFW-derivation

- Considering the contour integration $I = \oint dz A(z)/z$ by two ways:
  - Doing it along the point $z = \infty$, we get the "boundary contribution" $I = B$.
  - Doing it for big cycle around $z = 0$, we have $I = A(0) + \sum_\alpha \mathrm{Res}(A(z)/z)|_{z_\alpha}$.
- Combining above we have

$$A(z=0) = B - \sum_{poles\ z_\alpha} \mathrm{Res}\left(\frac{A(z)}{z}\right)_{z=z_\alpha}$$

Pole part

- Location: Pole happens when one propagator goes to on-shell, i.e., $P^2 + z(2P \cdot q) = 0$. From it we find the location of pole $z_\alpha = \frac{P_\alpha^2}{-2P \cdot q}$.
- Reside: Given by Factorization property:

$$\left(\frac{A(z)}{z}\right)_{z=z_\alpha} = \sum_\lambda A^L_{m+1}(1, .., m, P^\lambda(z_\alpha))$$

$$\frac{1}{P^2} A^R_{n-m+1}(-P^{-\lambda}(z_\alpha), m+1, ..., n)$$

Boundary part

- It has following three cases:
  - When $z \to \infty$, $A(z) \to \sum_{i=0}^{k} c_i z^i + \mathcal{O}(1/z)$ with $c_0 \neq 0 \Longrightarrow$ nonzero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z} \Longrightarrow$ zero boundary contribution
  - When $z \to \infty$, $A(z) \sim \frac{1}{z^k}$, $k \geq 2 \Longrightarrow$ zero boundary contribution and bonus relations
- Boundary behavior is a very nontrivial problem. Fortunately, for some theories under right choice of p_1, p_2, we have $\mathcal{M}(z) \to 0$ when $z \to \infty$. These include gauge and gravity theory.
  [Britto, Cachazo, Feng , Witten, 2004] [Arkani-Hamed, Kaplan 2008] [Cheung 2008]

BCFW recursion

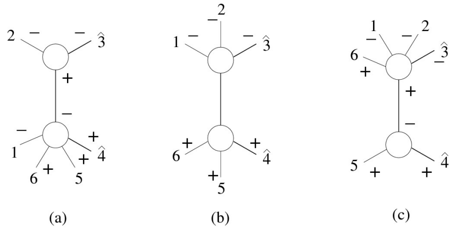
BCFW recursion relation for gluons:

[Britto, Cachazo, Feng , 2004]

- The formula is

$$A_n(1,2,\ldots,(n-1)^-,n^+) = \sum_{i=1}^{n-3}\sum_{h=+,-} A_{i+2}(\hat{n},1,2,\ldots i,-\hat{P}^h_{n,i}) \frac{1}{P^2_{n,i}} A_{n-i}(+\hat{P}^{-h}_{n,i},i+1,\ldots,n-2,n\hat{-}1)$$

- For 6-point, the contributed terms are given by



- The result is given by

$$A(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{1}{\langle 5|3+4|2]}$$

$$\left(\frac{\langle 1|2+3|4]^3}{[2\ 3][3\ 4]\ \langle 5\ 6\rangle\ \langle 6\ 1\rangle\ P^2_{234}} + \frac{\langle 3|4+5|6]^3}{[6\ 1][1\ 2]\ \langle 3\ 4\rangle\ \langle 4\ 5\rangle\ P^2_{345}} \right)$$

Let us work out the details of Figure (a).

- It is the product of two MHV amplitudes and a propagator,

$$\left(\frac{\left\langle 2\ \hat{3}\right\rangle^3}{\left\langle \hat{3}\ \hat{p}\right\rangle \left\langle \hat{p}\ 2\right\rangle}\right)\frac{1}{P_{23}^2}\left(\frac{\left\langle 1\ \hat{p}\right\rangle^3}{\left\langle \hat{p}\ \hat{4}\right\rangle\left\langle \hat{4}\ 5\right\rangle\left\langle 5\ 6\right\rangle\left\langle 6\ 1\right\rangle}\right) \tag{9}$$

- Note that

$$\lambda_{\hat{3}} = \lambda_3, \quad \lambda_{\hat{4}} = \lambda_4 - \frac{P_{23}^2}{\langle 3\ 2\rangle [2\ 4]}\lambda_3, \quad \langle \bullet\ \hat{p}\rangle = -\frac{\langle \bullet|2+3|4]}{[\hat{P}\ 4]} \tag{10}$$

(9) can straightforwardly be simplified to

$$\frac{\langle 1|2+3|4]^3}{[2\ 3][3\ 4]\langle 5\ 6\rangle\langle 6\ 1\rangle P_{234}^2\langle 5|3+4|2]}. \tag{11}$$

- Performing $i \to i+3$ and $\langle\ \rangle \leftrightarrow [\ ]$ in (11), we obtain Figure (c).

Generalization One—Massive theory

- The solution of q exists for $D \geq 4$. Thus it can be applied to massive theory and higher dimension quantum field theories
- For the case $p_j^2 \neq 0$, we first construct two null momenta by linear combinations $\eta_\pm = (p_i + x_\pm p_j)$ with $x_\pm = \left(-2p_i \cdot p_i \pm \sqrt{(2p_i \cdot p_j)^2 - 4p_i^2 p_j^2}\right) / 2p_j^2$. The solution can be

$$q = \lambda_{\eta_+} \widetilde{\lambda}_{\eta_-}, \qquad \text{or} \qquad q = \lambda_{\eta_-} \widetilde{\lambda}_{\eta_+} \ .$$

[Badger, Glover, Khoze and Svrcek, 2005]

Generalization Two— SUSY theory

- For $\mathcal{N} = 4$ theory, super-wave-function is given by Grassmann variables η^A $(A = 1, 2, 3, 4)$

$$\begin{aligned}\Phi(p,\eta) &= G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) \\ &+ \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^-(p)\,,\end{aligned}$$

- The generalized BCFW-deformation

$$\lambda_i(z) = \lambda_i + z\lambda_j, \quad \widetilde{\lambda}_j(z) = \widetilde{\lambda}_j - z\widetilde{\lambda}_j, \quad \eta_j(z) = \eta_j - z\eta_i,$$

so both momentum $\delta^4(\sum_i \lambda_i \widetilde{\lambda}_i)$ and super-momentum $\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A)$ conservations are kept

Generalization Two— SUSY theory

- Now we need to sum over super-multiplet

$$\mathcal{A} = \sum_{\text{split } \alpha} \int d^4\eta_{P_i} \mathcal{A}_L(p_i(z_\alpha), p_\alpha(z_\alpha)) \frac{1}{p_\alpha^2} \mathcal{A}_R(p_j(z_\alpha), -P_\alpha(z_\alpha)) \, .$$

[Arkani-Hamed, Cachazo and J. Kaplan, 2008; Brandhuber, Heslop and Travaglini, 2008]

Generalization Three– Off-shell current

- The famous Berends-Giele off-shell recursion relation is

$$J^{\mu}(1,2,...,k) = \frac{-i}{p_{1,k}^2}\left[\sum_{i=1}^{k-1} V_3^{\mu\nu\rho}(p_{1,i},p_{i+1,k}) J_{\nu}(1,...,i) J_{\rho}(i+1,...,k) + \sum_{j=i+1}^{k-1}\sum_{i=1}^{k-2} V_4^{\mu\nu\rho\sigma} J_{\nu}(1,...,i) J_{\rho}(i+1,...,j) J_{\sigma}(j+1,...,k)\right]$$

[Berends, Giele, 1988]

- Off-shell current is gauge dependent: (a) choice of polarization vector

$$\epsilon_{i\mu}^{+} = \frac{\langle r_i|\gamma_{\mu}|p_i]}{\sqrt{2}\langle r_i|p_i\rangle}, \qquad \epsilon_{i\mu}^{-} = \frac{[r_i|\gamma_{\mu}|p_i\rangle}{\sqrt{2}[r_i|p_i]}$$

Generalization Three– Off-shell current

- Gauge choice of propagator to fix to Feynman gauge
- To deal with the gauge dependence, we need to define two more polarization vectors

$$\epsilon_\mu^L = p_i, \quad \epsilon_\mu^T = \frac{\langle r_i|\gamma_\mu|r_i]}{2p_i \cdot r_i}$$

so we have

$$\begin{aligned} 0 &= \epsilon^+ \cdot \epsilon^+ = \epsilon^+ \cdot \epsilon^L = \epsilon^+ \cdot \epsilon^T = \epsilon^- \cdot \epsilon^- \\ &= \epsilon^- \cdot \epsilon^L = \epsilon^- \cdot \epsilon^T = \epsilon^T \cdot \epsilon^T = \epsilon^L \cdot \epsilon^L \\ 1 &= \epsilon^+ \cdot \epsilon^- = \epsilon^L \cdot \epsilon^T \end{aligned}$$

- The key observation is that now we have

$$g_{\mu\nu} = \epsilon_\mu^+ \epsilon_\nu^- + \epsilon_\mu^- \epsilon_\nu^+ + \epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L$$

Generalization Three– Off-shell current

- Taking $(i,j) = (1,k)$, the recursion relation is given by

$$\begin{aligned} & J^{\mu}(1,2,...,k) \\ = & \sum_{i=2}^{k-1} \sum_{h,\widetilde{h}} \left[A\left(\widehat{1},...,i,\widehat{p}^{h}\right) \cdot \frac{1}{p_{1,i}^{2}} \cdot J^{\mu}\left(-\widehat{p}^{\widetilde{h}}, i+1,...,\widehat{k}\right) \right. \\ & \left. + J^{\mu}\left(\widehat{1},...,i,\widehat{p}^{h}\right) \cdot \frac{1}{p_{i+1,k}^{2}} \cdot A\left(-\widehat{p}^{\widetilde{h}}, i+1,...,\widehat{k}\right) \right], \end{aligned}$$

where the sum is over $(h,\widetilde{h}) = (+,-),(-,+),(L,T),(T,L)$.

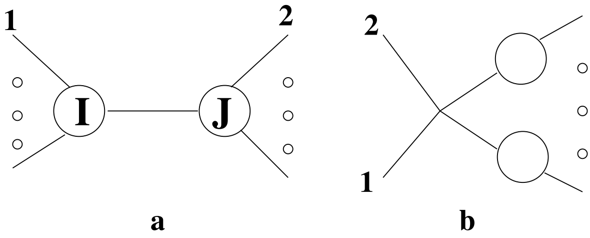
[Feng, Zhang, 2011]

Generalization Four– Nonzero boundary contribution

- Boundary is a (quasi)-global phenominon, i.e., depending the chosen pair and whole helicity configuration
- There are three ways to deal with boundary contributions:
  - Using auxiliary fields to make boundary zero [Benincasa, Cachazo, 2007; Boels, 2010]
  - Analyze Feynman diagrams directly [Feng, Wang, Wang, Zhang, 2009; Feng, Liu, 2010; Feng, Zhang, 2011]
  - Transfer to the discussion of roots of amplitude [Benincasa, Conde, 2011; Feng, Jia, Luo, Luo, 2011]
  - More pairs of deformation [Feng, Zhou, Qia, Rao, 2014]

Feynman diagram for $\lambda\phi^4$ theory

- With $(1,2)$-pair deformation, Feynman diagrams will be following two types:

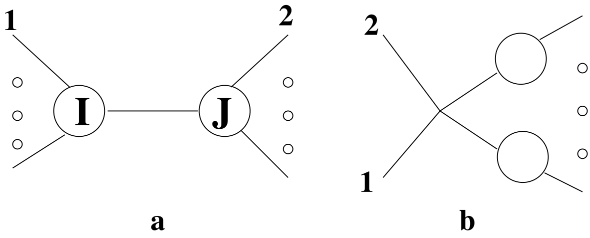


- Boundary contribution is

$$A_b = (-i\lambda) \sum_{\mathcal{I}'\bigcup\mathcal{J}'=\{n\}\backslash\{i,j\}} A_{\mathcal{I}'}(\{K_{\mathcal{I}'})) \frac{1}{p^2_{\mathcal{I}'}} \frac{1}{p^2_{\mathcal{J}'}} A_{\mathcal{J}'}(\{K_{\mathcal{J}'}\})$$

Feynman diagram for $\lambda\phi^4$ theory

- With $(1,2)$-pair deformation, Feynman diagrams will be following two types:

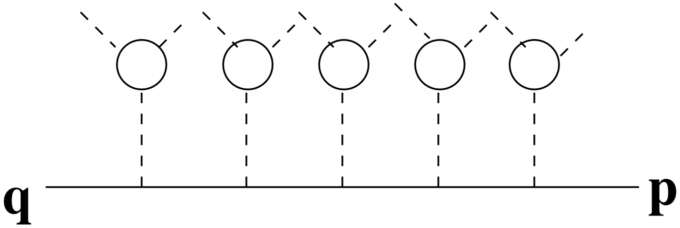


- Boundary contribution is

$$A_b = (-i\lambda) \sum_{\mathcal{I}' \bigcup \mathcal{J}' = \{n\} \backslash \{i,j\}} A_{\mathcal{I}'}(\{K_{\mathcal{I}'})) \frac{1}{p^2_{\mathcal{I}'}} \frac{1}{p^2_{\mathcal{J}'}} A_{\mathcal{J}'}(\{K_{\mathcal{J}'}\})$$

Feynman diagram for Yukawa theory

- Same analysis for typical Feynman diagram



- Only a few types of Feynman diagrams give boundary contributions and they can be evaluated directly

  [Feng, Wang, Wang, Zhang, 2009; Feng, Liu, 2010; Feng, Zhang, 2011]

Roots of amplitude

Another angel for boundary contributions:

- $$\begin{aligned} M_n(z) &= \sum_{k\in\mathcal{P}^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{p_k^2(z)} + C_0 + \sum_{l=1}^{v} C_l z^l \\ &= c\frac{\prod_s (z-w_s)^{m_s}}{\prod_{k=1}^{N_p} p_k^2(z)} \end{aligned}$$
- Split all roots into two groups $\mathcal{I},\mathcal{J}$. For $n_{\mathcal{I}} < N_p$

$$c\frac{\prod_{s=1}^{n_{\mathcal{I}}}(z-w_s)}{\prod_{k=1}^{N_p} p_k^2(z)} = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)}$$

$$M_n(z) = \sum_{k\in\mathcal{P}^{(i,j)}} \frac{c_k}{p_k^2(z)} \prod_{t=1}^{n_{\mathcal{J}}}(z-w_t)$$

Roots of amplitude

- Perform a contour integration around the pole z_k and obtain

$$\frac{M_L(z_k)M_R(z_k)}{(-2p_k\cdot q)}=\frac{c_k}{(-2p_k\cdot q)}\prod_{t=1}^{n_{\mathcal{J}}}(z_k-w_t),$$

so

$$c_k=\frac{M_L(z_k)M_R(z_k)}{\prod_{t=1}^{n_{\mathcal{J}}}(z_k-w_t)}$$

and finally

$$M_n(z)=\sum_{k\in\mathcal{P}^{(i,j)}}\frac{M_L(z_k)M_R(z_k)}{p_k^2(z)}\prod_{t=1}^{v+1}\frac{(z-w_t)}{z_k-w_t}$$

by setting $n_{\mathcal{I}}=N_p-1$.

[Benincasa, Conde, 2011; Feng, Jia, Luo, Luo, 2011]

Comments for boundary BCFW-relation

- Root method is very general and useful for theoretical discussions. However, it is very hard to find root recursively, especially roots are in general not rational function
- Feynman diagram method is practical, but not general since we need to do analysis for each different theory
- Both methods are not completely satisfied and better method is the more deformations!

Generalization five–Bonus relation

- Bonus relations can be derived from the observation

$$0=\oint \frac{dz}{z}z^b A(z),\quad b=1,1,...,a-1,\quad if,\ \ A(z)\to\frac{1}{z^a}\ .$$

Because the z^b factor, there is no pole at $z=0$. Taking contributions from other poles, we have bonus relations

$$0=\sum_{\alpha}\sum_{h}A_L(p^h(z_\alpha))\frac{z_\alpha^b}{p^2}A_R(-p^{-h}(z_\alpha))$$

for $b=1,...,a-1$.

[Arkani-Hamed, Cachazo, Kaplan, 2008]

Generalization six– Rational part of one loop amplitude

The new features appeared in this generalization are:

- There are double poles like $\langle a|b\rangle / [a|b]^2$, thus we need to find way to reproduce double pole and single pole contained inside double pole
- Loop factorization formula is

$$A_n^{1-\text{loop}} \to A_L^{1-\text{loop}} A_R^{\text{tree}} + A_L^{\text{tree}} A_R^{1-\text{loop}} + A_L^{\text{tree}} \mathcal{S} A_R^{\text{tree}} .$$

[Bern, Dixon, Kosower, 2005]

Generalization six– Rational part of one loop amplitude

Solution for above two difficulties:

- Two collinear momenta provide following divergent expression

$$A_{3;1}(1^+,2^+,3^+) = \frac{[1|2]\,[2|3]\,[3|1]}{K_{12}^2}$$

Thus double pole structure can then be obtained as

$$A_L^{\rm tree}\frac{1}{K_{a,a+1}^2}A_{3;1}(-\widehat{K}^+_{a,a+1},\widetilde{a}^+,(a+1)^+)$$
$$\to A_L^{\rm tree}\frac{1}{K_{a,a+1}^2}\frac{1}{K_{a,a+1}^2}\left[\widehat{K}_{a,a+1}|\widehat{a}\right]\left[\widehat{a}|a+1\right]\left[a+1|\widehat{K}_{a,a+1}\right]$$

[Bern, Dixon, Kosower, 2005]

Generalization six– Rational part of one loop amplitude

- Single pole inside double pole is solved by multiplying by a dimensionless function

$$K_{cd}^2 \mathcal{S}^{(0)}(a, s^+, b)\mathcal{S}^{(0)}(c, s^-, d)$$

, where the *soft factor* is given

$$\mathcal{S}^{(0)}(a, s^+, b) = \frac{\langle a|b\rangle}{\langle a|s\rangle \langle s|b\rangle}, \quad \mathcal{S}^{(0)}(c, s^-, d) = -\frac{[c|d]}{[c|s]\,[s|d]}\ .$$

[Bern, Dixon, Kosower, 2005]

Generalization seven– QFT in 3D

- The deformation null momentum q has solution when and only when $D \geq 4$.
- For 3D,

$$p^{\alpha\beta} = x^{\mu}(\sigma_{\mu})^{\alpha\beta} = \lambda^{\alpha}\lambda^{\beta}$$

thus on-shell BCFW-deformation can considered as matrix transformation over two spinors

$$\begin{pmatrix} \lambda_i(z) \\ \lambda_j(z) \end{pmatrix} = R(z) \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix},$$

This transformation keeps on-shell condition automatically

[Gang, Huang, Koh, Lee, Lipstein, 2010]

Generalization seven– QFT in 3D

- Conservation of momenta leads to

$$\begin{pmatrix} \lambda_i(z) & \lambda_j(z) \end{pmatrix} \begin{pmatrix} \lambda_i(z) \\ \lambda_j(z) \end{pmatrix} = \begin{pmatrix} \lambda_i & \lambda_j \end{pmatrix} \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix}$$

or

$$R^T(z)R(z) = I, \quad R(z) \in SO(2, C)$$

- With parameterization

$$R(z) = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix},$$

propagator is

$$\widehat{p}_f^2(z) = a_f z^{-2} + b_f + c_f z^2$$

Generalization seven– QFT in 3D

- Now the derivation is to start from contour integration

$$A(z=1) = \oint_{z=1} \frac{dz}{z-1} A(z)$$

where the contour is a small circle around $z=1$.

- Each on-shell propagator will gives four poles and we need to sum up their contributions.

Generalization eight– Different deformation

- Previous recursion relations based on the BCFW-deformation where two particles have been deformed
- However, there are other deformations we can consider. For example, for NMHV-amplitude, we do following holomorphic deformations

$$|i(z)] = |i] + z\langle j|k\rangle |\eta], \quad |j(z)] = |j] + z\langle k|i\rangle |\eta], \\ |k(z)] = |k] + z\langle i|j\rangle |\eta],$$

  where i, j, k have negative helicities.
- This deformation keeps (1) on-shell conditions; (2) momentum conservation.

[Risager, 2005]

Generalization eight– Different deformation

- Using the new deformation we can derive recursion relation using $\oint (dz/z)A(z)$ as

$$A = \sum_{\alpha, i\in A_L} A_L(z_\alpha)\frac{1}{p_\alpha^2}A_R(z_\alpha)$$

  It is nothing, but the MHV-decomposition for NMHV-amplitude.

  [Cachazo, Svrcek, Witten, 2004]

- For general $\mathrm{N}^{n-1}\mathrm{MHV}$-amplitudes, we make the deformation

$$|m_i(z)] = m_i + zr_i\,|\eta]\,, \quad i = 1, ..., n+1,$$

  for $n+1$ particles of negative helicity. Here $\sum_i r_i\,|m_i\rangle = 0$ to ensure momentum conservation.

  [Risager, 2005]

Part III : Applications of on-shell (BCFW) recursion relation

Applications of on-shell recursion relation can be divided into following two types:

- **Calculation of various amplitudes:** This is the initial motivation leading to the discovery of on-shell recursion relation. It is also one of most important practical applications for high energy experiments.
- **Understanding of various properties of QFT:** It has two distinguish features:
  - It keeps only on-shell information
  - It relies only on some general properties of QFT, so it opens new way to study QFT in the frame of S-matrix program

Contents

On-shell plus S-matrix

Now we discuss some general properties for massless quantum fields, which is initiated by Benincase and Cachazo. [Benincase, Cachazo, 2007]

For massless field, we have

- On-shell condition makes the better variable to be spinor λ and anti-spinor $\widetilde{\lambda}$ (not the familiar p_μ and polarization vector)
- Lorentz invariance allows only following combinations $\langle\lambda_1|\lambda_2\rangle$ and $\left[\widetilde{\lambda}_1|\widetilde{\lambda}_2\right]$
- λ carries helicity charge $-1/2$ and $\widetilde{\lambda}$ carries helicity charge $+1/2$. Amplitude must carry the right helicity for each massless particle

On-shell plus S-matrix

First nontrivial result:

- Three-point on-shell amplitude is completely determined by Lorentz symmetry and helicity. For example, with the case $h_1 = h_2 = h_3$ we have following four configurations:

$$
\begin{aligned}
M_3(1^-_m, 2^-_r, 3^+_s) &= \kappa_{mrs} \left(\frac{\langle 1|2\rangle^3}{\langle 2|3\rangle \langle 3|1\rangle} \right)^h , \\
M_3(1^+_m, 2^+_r, 3^-_s) &= \kappa_{mrs} \left(\frac{[1|2]^3}{[2|3]\,[3|1]} \right)^h , \\
M_3(1^-_m, 2^-_r, 3^-_s) &= \widetilde{\kappa}_{mrs} \left(\langle 1|2\rangle \langle 2|3\rangle \langle 3|1\rangle \right)^h , \\
M_3(1^+_m, 2^+_r, 3^+_s) &= \widetilde{\kappa}_{mrs} \left([1|2]\,[2|3]\,[3|1] \right)^h ,
\end{aligned}
$$

- Two immediately conclusions:
  - By crossing symmetry, when $h = odd$, the κ_{mrs} must to totally antisymmetric, while when $h = even$, the κ_{mrs} must to totally symmetric.
  - Thus for vector with $h = 1$, if we have less than three particle types, it is identical zero. It is very familiar for $U(1)$ photo without self-interaction.
  - To have self-interaction, the minimal number is three, i.e., the non-Abelian $SU(2)$.

On-shell plus S-matrix

Second nontrivial result: Four particle test

- Now we consider $A(1,2,3,4)$ with $h_i = 1$ and calculate it using two different deformations: (1) $(1,2)$-deformation; (2) $(4,1)$-deformation.
- Results from both deformations should be same. The consistent condition gives familiar Jacobi identity

$$\sum_{a_I} f_{a_1 a_4 a_I} f_{a_I a_3 a_2} + \sum_{a_I} f_{a_1 a_3 a_I} f_{a_I a_4 a_2} + \sum_{a_I} f_{a_1 a_2 a_I} f_{a_I a_3 a_4} = 0.$$

On-shell plus S-matrix

- If all particles with $h_i = 2$, four particle test tell us that the algebra defined by

$$\mathcal{E}_a \star \mathcal{E}_b = f_{abc}\, \mathcal{E}_c$$

  must be commutative and associative.
- With $h > 2$ we can show there is no non-trivial way to satisfy the four-particle test
- When some particle with $h = 2$ and some with $h < 2$, coupling constant should be same.

Contents

Now we want to show following four facts:

- (1) Color-order reversed relation for general n;
- (2) The $U(1)$-decoupling relation;
- (3) The KK-relation;
- (4) The BCJ relation;

The only assumption we will use: BCFW cut-constructibility of gluon amplitudes

Another fact from previous discussion is that for color-ordered three-point amplitude we have

$$A(1,2,3) = -A(3,2,1)$$

[Feng, Huang, Jia, 2010]

Color-order reversed relation

Color-order reversed relation:

$$\begin{aligned}
& A(1,n,\{\beta_1,...,\beta_{n-2}\}) \\
= & \sum_{i=1}^{n-3} A(n,\beta_1,...,\beta_i,-P)\frac{1}{P^2}A(P,\beta_{i+1},...,\beta_{n-2},1) \\
= & \sum_{i=1}^{n-3} (-)^{n-i}A(1,\beta_{n-2},...,\beta_{i+1},P)\frac{1}{P^2}(-)^{i+2}A(-P,\beta_i,...,\beta_1,n) \\
= & (-)^n A(1,\beta_{n-2},\beta_{n-1},...,\beta_1,n)
\end{aligned}$$

$U(1)$-decoupling identity

$U(1)$-decoupling identity. It can be done by induction for which we use $n = 5$ to show the idea (by $(1, 2)$-deformation)

$$\begin{array}{lclclclcl}
A(1,2,3,4,5) & = & A(1,P_{23},4,5) & + & A(1,P_{234},5) & + & 0 \\
A(1,5,2,3,4) & = & A(1,5,P_{23},4) & + & A(1,5,P_{234}) & + & A(1,P_{52},3,4) \\
A(1,4,5,2,3) & = & A(1,4,5,P_{23}) & + & 0 & + & A(1,4,P_{52},3) \\
A(1,3,4,5,2) & = & 0 & + & 0 & + & A(1,3,4,P_{52})
\end{array}$$

$$\begin{array}{clcl}
+ & 0 & + & 0 \\
+ & A(1,P_{523},4) & + & 0 \\
+ & A(1,4,P_{523}) & + & A(1,P_{452},3) \\
+ & 0 & + & A(1,3,P_{452})
\end{array}$$

where

$$A(1,P_{23},4,5) \equiv A(\widehat{1},\widehat{P}_{23},4,5)\frac{1}{s_{23}}A(-\widehat{P}_{23},\widehat{2},3)$$

KK-relation

- KK-relation: [Kleiss, Kujif, 1989]

$$A_n(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in OP(\{\alpha\},\{\beta^T\})} A_n(1,\sigma,n) \ .$$

where sum is over partial ordering.

- Example

$$A(1,\{2\},5,\{3,4\} = A(1,2,4,3,5)$$
$$+A((1,4,2,3,5) + A(1,4,3,2,5)$$

KK-relation

- We use $(1,5)$-shifting for $n=5$. First step we do BCFW expansion:

$$
\begin{aligned}
A(1,2,5,3,4) &= A(4,1,2,P_{35})\frac{1}{P_{35}^2}A(-P_{35},5,3) \\
&+A(3,4,1,P_{25})\frac{1}{P_{25}^2}A(-P_{25},2,5) \\
&+A(1,2,-P_{12})\frac{1}{P_{12}^2}A(P_{12},5,3,4) \\
&+A(4,1,-P_{41})\frac{1}{P_{41}^2}A(P_{41},2,5,3)
\end{aligned}
$$

Using Color-order reverse, $U(1)$ and KK for components:

$$\begin{aligned}
& A(1,2,5,3,4) \\
= \quad & (-A(1,2,4,P_{35}) - A(1,4,2,P_{35}))\frac{1}{P_{35}^2}(-A(-P_{35},3,5)) \\
& +A(1,4,3,P_{25})\frac{1}{P_{25}^2}A(-P_{25},2,5) \\
& +A(1,2,-P_{12})\frac{1}{P_{12}^2}A(P_{12},4,3,5) \\
& +(-A(1,4,-P_{41}))\frac{1}{P_{41}^2}(-A(P_{41},2,3,5) - A(P_{41},3,2,5))
\end{aligned}$$

$$T_1 + T_4 = A(1,2,4,3,5), \quad T_2 + T_5 = A(1,4,2,3,5), \\ T_3 + T_6 = A(1,4,3,2,5)$$

BCJ relation

- First we want to remark that all BCJ-relation can be derived from the one with length one at the set α. We call it the fundamental set.
- The form of fundamental one

$$\begin{aligned}
0 &= I_4 = A(2,4,3,1)(s_{43}+s_{41}) + A(2,3,4,1)s_{41} \\
0 &= I_5 = A(2,4,3,5,1)(s_{43}+s_{45}+s_{41}) \\
&\quad + A(2,3,4,5,1)(s_{45}+s_{41}) + A(2,3,5,4,1)s_{41} \\
0 &= I_6 = A(2,4,3,5,6,1)(s_{43}+s_{45}+s_{46}+s_{41}) \\
&\quad + A(2,3,4,5,6,1)(s_{45}+s_{46}+s_{41}) \\
&\quad + A(2,3,5,4,6,1)(s_{46}+s_{41}) + A(2,3,5,6,4,1)s_{41}
\end{aligned}$$

- The dual format by momentum conservation

$$\begin{aligned} 0 &= A(2,4,3,5,1)s_{24} + A(2,3,4,5,1)(s_{24}+s_{34}) \\ &\quad + A(2,3,5,4,1)(s_{24}+s_{34}+s_{54}) \end{aligned}$$

- A special case with $n = 3$: $A(1,2,3)s_{23} = 0$.

BCJ relation

- Take $(1,6)$ to do the deformation, consider combination

$$\begin{aligned} I_6(z) &= s_{2\hat{1}}A(\hat{1},2,3,4,5,\hat{6}) + (s_{2\hat{1}} + s_{32})A(\hat{1},3,2,4,5,\hat{6}) \\ &+(s_{2\hat{1}} + s_{32} + s_{42})A(\hat{1},3,4,2,5,\hat{6}) \\ &+(s_{2\hat{1}} + s_{32} + s_{42} + s_{52})A(\hat{1},3,4,5,2,\hat{6}) \end{aligned}$$

- Consider contour integration $\oint_{z=0} \frac{dz}{z} I_6(z) = I_6(z=0)$.
- Same contour can be evaluated using the finite poles plus boundary contribution.

- To see boundary part,

$$\begin{aligned} I_6(z) &= I_1 + I_2 \\ I_1 &= s_{2\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6}) + A(\widehat{1},3,2,4,5,\widehat{6}) \\ &\quad +A(\widehat{1},3,4,2,5,\widehat{6}) + A(\widehat{1},3,4,5,2,\widehat{6})\Big] \\ &= -s_{2\widehat{1}}A(\widehat{1},3,4,5,\widehat{6},2) \to \frac{1}{z} \end{aligned}$$

where KK-relation has been used, while for

$$I_2 \to \frac{1}{z}$$

- Result $\oint_{z=\infty}\frac{dz}{z}I_6(z) = 0$

Finite pole part, expansion by on-shell recursion relation

- $$\begin{aligned} A(\widehat{1},2,3,4,5,\widehat{6}) &\to & s_{2\widehat{1}}A_3(\widehat{1},2,P) & \quad A(-P,3,4,5,\widehat{6}) \\ A(\widehat{1},3,2,4,5,\widehat{6}) &\to & -A_3(\widehat{1},3,P) & \quad A(-P,2,4,5,\widehat{6})(s_{24}+s_{25}+s_{2\widehat{6}}) \\ A(\widehat{1},3,4,2,5,\widehat{6}) &\to & -A_3(\widehat{1},3,P) & \quad A(-P,4,2,5,\widehat{6})(s_{25}+s_{2\widehat{6}}) \\ A(\widehat{1},3,4,5,2,\widehat{6}) &\to & -A_3(\widehat{1},3,P) & \quad A(-P,4,5,2,\widehat{6})(s_{2\widehat{6}}) \end{aligned}$$

- $$\begin{aligned} A(\widehat{1},2,3,4,5,\widehat{6}) &\to & s_{2\widehat{1}}A_3(\widehat{1},2,3,P) & \quad A(-P,4,5,\widehat{6}) \\ A(\widehat{1},3,2,4,5,\widehat{6}) &\to & (s_{2\widehat{1}}+s_{23})A_3(\widehat{1},3,2,P) & \quad A(-P,4,5,\widehat{6}) \\ A(\widehat{1},3,4,2,5,\widehat{6}) &\to & -A_3(\widehat{1},3,4,P) & \quad A(-P,2,5,\widehat{6})(s_{25}+s_{2\widehat{6}} \\ A(\widehat{1},3,4,5,2,\widehat{6}) &\to & -A_3(\widehat{1},3,4,P) & \quad A(-P,5,2,\widehat{6})(s_{2\widehat{6}}) \end{aligned}$$

- $$\begin{aligned}
A(\widehat{1},2,3,4,5,\widehat{6}) &\to & s_{2\widehat{1}} A_3(\widehat{1},2,3,4,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,2,4,5,\widehat{6}) &\to & (s_{2\widehat{1}}+s_{23}) A_3(\widehat{1},3,2,4,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,4,2,5,\widehat{6}) &\to & (s_{2\widehat{1}}+s_{23}+s_{24}) A_3(\widehat{1},3,4,2,P) & \quad A(-P,5,\widehat{6}) \\
A(\widehat{1},3,4,5,2,\widehat{6}) &\to & -A_3(\widehat{1},3,4,5,P) & \quad A(-P,2,\widehat{6})(s_{2\widehat{6}}
\end{aligned}$$

- For the general n, the proof will be exactly same

Contents

The differences between gauge theory and gravity

Let us compare gauge theory and gravity theory:

- Gauge symmetry is symmetry for inner quantities while gravity theory is based on the space-time symmetry, the general equivalence principal for the choice of coordinate.
- The spin of gauge bosons is one while the spin of graviton is two.
- More importantly, the Lagrangian of gauge theory is polynomial with finite interaction terms while the Einstein Lagrangian is highly non-linear and infinite interaction terms after perturbative expansion.

However, we must be careful about these differences we have talked:

- The Lagrangian description is a off-shell description. What happens if we constraint to only on-shell quantities?
- We have clues from string theory:
    - Graviton given by closed string; Gluons given by open string.
    - Closed string $===$ left-moving open mode $\times$ right moving open mode
    - In one word, on-shell Graviton $==$ [Gluon]^2

- One accurate description of above claim is the KLT relation for tree-level scattering amplitude, which is obtained from string theory. For example

$$\begin{aligned} \mathcal{M}_3(1,2,3) &= A_3(1,2,3)\widetilde{A}_3(1,2,3), \\ \mathcal{M}_4(1,2,3,4) &= A_4(1,2,3,4)s_{12}\widetilde{A}_4(3,4,2,1) \end{aligned}$$

[Kawai, Lewellen, Tye; 1985] [Bern, Dixon, Perelstein, Rozowsky; 1999]

- Question: Could we understand this relation directly in the framework of quantum field theory?

Idea of field theory proof of KLT

Now we can give the idea of field theory proof of KLT relation:

- First using only the Lorentz invariance and spin symmetry we have $\mathcal{M}_3(1,2,3) = A_3(1,2,3)\widetilde{A}_3(1,2,3)$.
- Using BCFW-relation to expand gluon amplitudes and then recombine them to give the BCFW expansion of graviton amplitude. Thus by the induction method, we have the pure field theory proof.

Example One: four gravitons with relation

$$M_4(1,2,3,4) \;=\; (-)s_{12}A(1,3,4,2)A(1,4,3,2)$$

- Step one: Using $(1,2)$-BCFW-shifting to make

$$I \;=\; \oint \frac{dz}{z}(-)s_{12}A(\widehat{1},3,4,\widehat{2})A(\widehat{1},4,3,\widehat{2}) = 0$$

- BCFW expansion to get

$$\sum_h s_{12}A_3(\widehat{1},3,-\widehat{P}^h_{13})\frac{1}{s_{13}}A_3(\widehat{P}^{-h}_{13},4,\widehat{2})A_4(\widehat{1}(z_{13}),4,3,\widehat{2}(z_{13}))$$

$$+\;\sum_h s_{12}A_4(\widehat{1}(z_{14}),3,4,\widehat{2}(z_{14}))A_3(\widehat{1},4,-\widehat{P}^h_{14})\frac{1}{s_{14}}A_3(\widehat{P}^h_{14},3,\widehat{2})$$

- For the first line we can use the BCJ relation $s_{12}A_4(\widehat{1}(z_{13}),4,3,\widehat{2}(z_{13}))=s_{13}(z_{13})A_4(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})$ to write it as

$$A_3(\widehat{1},3,-\widehat{P}_{13}^h)\frac{1}{s_{13}}A_3(\widehat{P}_{13}^{-h},4,\widehat{2})s_{13}(z_{13})A_4(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})\,.$$

- Naively in the cut z_{13} we will have $s_{13}(z_{13})=0$. However, notice that

$$\begin{aligned} & A_4(4,2,3,1) \\ = & \sum_h \frac{A_3(4,\widehat{2}(z_{13}),\widehat{P}_{13}(z_{13}))A_3(-\widehat{P}_{13}(z_{13}),3,\widehat{1}(z_{13}))}{s_{13}} \\ & +\frac{A_3(\widehat{1},4,\widehat{P}_{23}(z_{13})A_3(-\widehat{P}_{23}(z_{14}),3,\widehat{1}(z_{14}))}{s_{14}} \end{aligned}$$

- Thus we see that

$$
\begin{aligned}
& s_{13}(z_{13})A(4,\widehat{2}(z_{13}),3,\widehat{1}(z_{13})) \\
& = \sum_h A(4,\widehat{2}(z_{13}),P_{23})A(-P_{23}(z_{13}),3,\widehat{1}(z_{13}))
\end{aligned}
$$

- Doing similarly for the second term we obtain

$$
\begin{aligned}
& \sum_{h,\widetilde{h}} A_3(\widehat{1},3,-\widehat{P}_{13}^{h})\frac{1}{s_{13}}A_3(\widehat{P}_{13}^{-h},4,\widehat{2})A_3(\widehat{1},3,-\widehat{P}_{13}^{\widetilde{h}})A_3(\widehat{P}_{13}^{-\widetilde{h}},4,\widehat{2}) \\
+ & \sum_{h,\widetilde{h}} A_3(\widehat{1},4,-\widehat{P}_{14}^{\widetilde{h}})A_3(\widehat{P}_{14}^{-\widetilde{h}},3,\widehat{2})A_3(\widehat{1},4,-\widehat{P}_{14}^{h})\frac{1}{s_{14}}A_3(\widehat{P}_{14}^{-h},3,\widehat{2})
\end{aligned}
$$

- The double sum $\sum_{h,\widetilde{h}}$ can be written as two sums $\sum_{\widetilde{h}=h}$ and $\sum_{\widetilde{h}=-h}$.
- We have also vanishing identity for flipped helicity

$$A_3(\widehat{1},3,-\widehat{P}_{13}^{+})A_3(\widehat{1},3,-\widehat{P}_{13}^{-})=0\ .$$

- Using three point result we can combine to get

$$M_4(1,2,3,4)=\sum_{h=+,-}M_3(\widehat{1},3,-\widehat{P}_{13}^{h})\frac{1}{s_{13}}M_3(\widehat{P}_{13}^{-h},4,\widehat{2})$$
$$+M_3(\widehat{1},4,-\widehat{P}_{14}^{h})\frac{1}{s_{14}}M_3(\widehat{P}_{14}^{-h},3,\widehat{2})$$

Function S:

- To write down the general KLT relation, we need following function

$$\mathcal{S}[i_1, \ldots, i_k | j_1, j_2, \ldots, j_k]_{p_1} = \prod_{t=1}^{k} (s_{i_t 1} + \sum_{q>t}^{k} \theta(i_t, i_q) s_{i_t i_q})$$

where $\theta(i_t, i_q) = 0$ is zero when pair (i_t, i_q) has same ordering at both set $\mathcal{I}, \mathcal{J}$ and otherwise, it is one.. Set $\mathcal{J}$ is the reference ordering set.

$$\begin{aligned} \mathcal{S}[2,3,4|2,4,3] &= s_{21}(s_{31} + s_{34})s_{41}, \\ \mathcal{S}[2,3,4|4,3,2] &= (s_{21} + s_{23} + s_{24})(s_{31} + s_{34})s_{41} \end{aligned}$$

- Property:

$$\mathcal{S}[i_1,...,i_k|j_1,j_2,...,j_k] = \mathcal{S}[j_k,...,j_1|i_k,..,i_1]$$

- Dual function

$$\widetilde{\mathcal{S}}[i_2,..,i_{n-1}|j_2,...,j_{n-1}]_{p_n} = \prod_{t=2}^{n-1}(s_{j_t n} + \sum_{q<t}\theta(j_t,j_q)s_{j_t j_q}) \ .$$

$\widetilde{\mathcal{S}}$ and $\mathcal{S}$ are related as follows:

$$\widetilde{\mathcal{S}}[\mathcal{I}|\mathcal{J}]_{p_n} = \mathcal{S}[\mathcal{J}^T|\mathcal{I}^T]_{p_n}$$

$$\widetilde{\mathcal{S}}[2,3,4|4,3,2] = s_{45}(s_{35}+s_{34})(s_{25}+s_{23}+s_{24})$$

- A crucial property

$$I = \sum_{\alpha \in S_k} \mathcal{S}[\alpha(i_1, ..., i_k)|j_1, j_2, ..., j_k] A(k+2, \alpha(i_1, ..., i_k), 1) = 0$$

by BCJ relation.

General KLT relations:

- The manifest $(n-3)!$ symmetric form

$$
\begin{aligned}
& M_n \\
= \; & (-)^{n+1} \sum_{\sigma\in S_{n-3}} \sum_{\alpha\in S_j} \sum_{\beta\in S_{n-3-j}} A(1,\{\sigma_2,..,\sigma_j\},\{\sigma_{j+1},..,\sigma_{n-2}\},n-1,n) \\
& \mathcal{S}[\alpha(\sigma_2,..,\sigma_j)|\sigma_2,..,\sigma_j]_{p_1} \widetilde{\mathcal{S}}[\sigma_{j+1},..,\sigma_{n-2}|\beta(\sigma_{j+1},..,\sigma_{n-2})]_{p_{n-1}} \\
& \widetilde{A}(\alpha(\sigma_2,..,\sigma_j),1,n-1,\beta(\sigma_{j+1},..,\sigma_{n-2}),n)
\end{aligned}
$$

[Bern, Dixon, Perelstein, Rozowsky; 1999]

- The set I or set J can be empty, so we have two more symmetric forms:

$$M_n = (-)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A(1,\sigma(2,n-2),n-1,n)$$
$$\mathcal{S}[\widetilde{\sigma}(2,n-2))|\sigma(2,n-2))]_{p_1} \widetilde{A}(n-1,n,\widetilde{\sigma}(2,n-2),1)$$

as well as

$$M_n = (-)^{n+1} \sum_{\sigma,\widetilde{\sigma}\in S_{n-3}} A(1,\sigma(2,n-2),n-1,n)$$
$$\widetilde{\mathcal{S}}[\sigma(2,n-2))|\widetilde{\sigma}(2,n-2))]_{p_{n-1}} \widetilde{A}(1,n-1,\widetilde{\sigma}(2,n-2),n)$$

The $(n-2)!$ symmetric new KLT formula:

$$M_n = (-)^n \sum_{\gamma,\beta} \widetilde{A}(n, \gamma(2, ..., n-1), 1)$$
$$\mathcal{S}[\gamma(2, ..., n-1)|\beta(2, .., n-1)]_{p_1} A(1, \beta(2, ..., n-1), n)/s_{123..(n-1)}$$

and

$$M_n = (-)^n \sum_{\beta,\gamma} A(1, \beta(2, ..., n-1), n)$$
$$\widetilde{\mathcal{S}}[\beta(2, ..., n-1)|\gamma(2, .., n-1)]_{p_n} \widetilde{A}(n, \gamma(2, ..., n-1), 1)/s_{2...n}$$

New vanishing identities:

If we use the (n_+, n_-) to denote the number of positive (negative) helicities in A having been flipped in $\widetilde{A}$, then when $n_+ \neq n_-$, we obtain zero, i.e.,

$$0 = (-)^n \sum_{\gamma,\beta} \widetilde{A}_{n_+ \neq n_-}(n, \gamma(2, ..., n-1), 1)$$
$$\mathcal{S}[\gamma(2, ..., n-1)|\beta(2, .., n-1)]_{p_1} A(1, \beta(2, ..., n-1), n)/s_{123..(n-1)}$$

The BCFW proof of the new KLT formula: First step, the pole structure analysis of a general one, for example, $s_{12..k}$

- The pole appears in only one of the amplitudes $\widetilde{A}_n$ and A_n.
- The pole appears in both amplitudes $\widetilde{A}_n$ and A_n.

The BCFW proof of the new KLT formula: First step, the pole structure analysis of a general one, for example, $s_{12..k}$

- The pole appears in only one of the amplitudes $\widetilde{A}_n$ and A_n.
- The pole appears in both amplitudes $\widetilde{A}_n$ and A_n.

The second step is to show the structure (A) giving zero:

- The BCFW expansion is given by

$$\frac{(-1)^{n+1}}{s_{\widehat{12..n-1}}}\sum_{\gamma,\sigma,\beta}\frac{\sum_h \widetilde{A}_{n-k+1}(\widehat{n},\gamma,-\widehat{P}^h)\widetilde{A}_{k+1}(\widehat{P}^{-h},\sigma,\widehat{1})}{s_{12..k}}$$
$$\times\mathcal{S}[\gamma\sigma|\beta_{2,..,n-1}]A_n(\widehat{1},\beta_{2,..,n-1},\widehat{n}),$$

- Important observation:

$$\mathcal{S}[\gamma\sigma|\beta_{2,..,n-1}]=\mathcal{S}[\sigma|\rho_{2,k}]\times(\text{a factor independent of }\sigma),$$

- By BCJ relation

$$\sum_{\sigma}\widetilde{A}_{k+1}(\widehat{P}^{-h},\sigma,\widehat{1})\mathcal{S}[\sigma|\rho_{2,k}]=0,$$

The third step is to show the part (B) giving the desired result:

- The BCFW expansion is now

$$\frac{(-1)^{n+1}}{s_{\widehat{1}2\ldots(n-1)}}\sum_{\gamma,\beta,\sigma,\alpha}\left[\frac{\sum_h \widetilde{A}(\widehat{n},\gamma,\widehat{P}^{-h})\widetilde{A}(-\widehat{P}^{h},\sigma,\widehat{1})}{s_{12..k}}\right]\mathcal{S}[\gamma\sigma|\alpha\beta]$$
$$\left[\frac{\sum_h A(\widehat{1},\alpha,-\widehat{P}^{h})A(\widehat{P}^{-h},\beta,\widehat{n})}{s_{\widehat{1}2..k}}\right],$$

- Using $\mathcal{S}[\gamma\sigma|\alpha\beta] = \mathcal{S}[\sigma|\alpha] \times \mathcal{S}_{\widehat{P}}[\gamma|\beta]$ we obtain

$$\frac{(-1)^{n+1}}{s_{12..k}} \sum_h \left[\left(\sum_{\sigma,\alpha} \frac{\widetilde{A}(-\widehat{P}^h, \sigma, \widehat{1}) \mathcal{S}[\sigma|\alpha] A(\widehat{1}, \alpha, -\widehat{P}^h)}{s_{\widehat{1}2..k}} \right) \left(\sum_{\gamma,\beta} \frac{\widetilde{A}(\widehat{n}, \gamma, \widehat{P}^{-h}) \mathcal{S}_{\widehat{P}}[\gamma|\beta] A(\widehat{P}^{-h}, \beta, \widehat{n})}{s_{\widehat{P}k+1..(n-1)}} \right) \right] + (h, -h),$$

- It is nothing but

$$-\frac{\sum_h M_{k+1}(\widehat{1}, 2, \ldots, k, -\widehat{P}^h) M_{n-k+1}(\widehat{P}^{-h}, k+1, \ldots, \widehat{n})}{s_{12..k}},$$

The proof of $(n-3)!$ form will be almost same:

- Divide the pole structure into (A) and (B) part.
- Using the BCJ to show the (A) part to be zero.
- Using the $(n-2)!$ form to show that the part (B) is nothing, but the BCFW expansion.

Some remarks

- The on-shell structure of $[gravity] = [gluon]^2$ is extremely important. One can apply it to construct the loop amplitude of SUGRV.
- The reason we have the simple proof is because on-shell recursion relation has got rid of complicated off-shell information

The study of analytic property of scattering amplitudes has caught many attentions in recent years. Although there are huge progresses we have made, there are still more waiting us to discover!