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The Neumann-to-Dirichlet map in two dimensions

O.Yu. Imanuvilov [a,1], Gunther Uhlmann [b,c,d,*,2], M. Yamamoto [e]

[a] Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins, CO 80523-1874, USA
[b] Department of Mathematics, University of Washington, Seattle, WA 98195, USA
[c] Department of Mathematics and Statistics, University of Helsinki, Finland
[d] HKUST Jockey Club Institute for Advanced Study, HKUST, Clear Water Bay, Kowloon, Hong Kong, China
[e] Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153, Japan

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A B S T R A C T

For the two-dimensional Schrödinger equation in a bounded domain, we prove uniqueness of the determination of potentials in $W_p^1(\Omega)$, $p > 2$ in the case where we apply all possible Neumann data supported on an arbitrarily non-empty open set $\widetilde{\Gamma}$ of the boundary and observe the corresponding Dirichlet data on $\widetilde{\Gamma}$. An immediate consequence is that one can uniquely determine a conductivity in $W_p^3(\Omega)$ with $p > 2$ by measuring the voltage on an open subset of the boundary corresponding to a current supported in the same set.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\nu = (\nu_1, \nu_2)$ be the unit outer normal to $\partial\Omega$ and let $\frac{\partial}{\partial\nu} = \nabla \cdot \nu$.

* Corresponding author.
E-mail addresses: oleg@math.colostate.edu (O.Yu. Imanuvilov), gunther@math.washington.edu (G. Uhlmann), myama@ms.u-tokyo.ac.jp (M. Yamamoto).
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In this domain we consider the Schrödinger equation with a potential q:

$$L_q(x, D)u = (\Delta + q)u = 0 \quad \text{in } \Omega. \tag{1}$$

Let $\widetilde{\Gamma}$ be a non-empty arbitrary fixed relatively open subset of $\partial\Omega$. Consider the Neumann-to-Dirichlet map $N_{q,\widetilde{\Gamma}}$ with partial data on $\widetilde{\Gamma}$ defined by

$$N_{q,\widetilde{\Gamma}} : f \to u|_{\widetilde{\Gamma}}, \tag{2}$$

where

$$(\Delta + q)u = 0 \quad \text{in } \Omega, \ \frac{\partial u}{\partial \nu}|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \ \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = f \tag{3}$$

with domain $D(N_{q,\widetilde{\Gamma}}) \subset L^2(\widetilde{\Gamma})$. Without loss of generality we may assume that $\partial\Omega \setminus \widetilde{\Gamma}$ contains a non-empty open set. By uniqueness of the Cauchy problem for the Schrödinger equation the operator $N_{q,\widetilde{\Gamma}}$ is well defined since the problem (3) has at most one solution for each $f \in L^2(\widetilde{\Gamma})$. Thanks to the Fredholm alternative, we see that $D(N_{q,\widetilde{\Gamma}}) = \overline{D(N_{q,\widetilde{\Gamma}})}$ and $L^2(\widetilde{\Gamma}) \setminus D(N_{q,\widetilde{\Gamma}})$ is finite dimensional for any potential q in $H^1(\Omega)$.

The goal of this article is to prove uniqueness of the determination of the potential q from the Neumann-to-Dirichlet map $N_{q,\widetilde{\Gamma}}$ given by (2) for arbitrary subboundary $\widetilde{\Gamma}$. More precisely, we consider all Neumann data supported on an arbitrarily fixed subboundary $\widetilde{\Gamma}$ as input and we observe the Dirichlet data only on the same subboundary $\widetilde{\Gamma}$. This map arises in electrical impedance tomography (EIT) where one attempts to determine the electrical conductivity of a medium by inputting voltages and measuring current at the boundary. After transforming (1) to the conductivity equation, we can interpret $u|_{\widetilde{\Gamma}}$ and $\frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}$ respectively as the voltage and the multiple of the current by values of the surface conductivity. In practice, we can realize such inputs and outputs by applying current to electrodes on the boundary and observing the corresponding voltages. The current inputs are modeled by the Neumann boundary data $\frac{\partial u}{\partial \nu}$ and the observation data are modeled by Dirichlet data. See e.g. Cheney, Issacson and Newell [11] for applications to medical imaging of EIT. Moreover it is very desirable to restrict the supports of the current inputs as small as possible. To the authors' best knowledge there are few works on the uniqueness by such a "Neumann-to-Dirichlet map" with partial data. In Astala, Päivärinta and Lassas [3], the authors consider both the Dirichlet-to-Neumann map and the Neumann-to-Dirichlet map on an arbitrarily subboundary to establish the uniqueness of an anisotropic conductivity modulo the group of diffeomorphisms which is the identity on the boundary where the measurements take place.

The case where the measurements are given by the Dirichlet-to-Neumann map has been extensively studied in the literature. This map is defined in the case of partial data by

$$\Lambda_{q,\widetilde{\Gamma}} : g \to \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}}; \quad (\Delta + q)u = 0 \quad \text{in } \Omega, \ u|_{\partial\Omega\setminus\widetilde{\Gamma}} = 0, \quad u|_{\widetilde{\Gamma}} = g.$$

We give some references but the list is not at all complete. In the case of full data $\widetilde{\Gamma} = \partial\Omega$, this inverse problem was formulated by Calderón [10]. In the two-dimensional case, given a Dirichlet-to-Neumann map $\Lambda_{q,\widetilde{\Gamma}}$ on an arbitrary subboundary $\widetilde{\Gamma}$, uniqueness is proved under the assumption $q \in C^{2+\alpha}(\overline{\Omega})$ by Imanuvilov, Uhlmann and Yamamoto [15] and for the uniqueness for potentials $q \in W_p^1(\Omega)$, $p > 2$ see Imanuvilov and Yamamoto [20]. For other uniqueness results by the Dirichlet-to-Neumann map on an arbitrary subboundary $\widetilde{\Gamma}$, we can refer also to Imanuvilov, Uhlmann and Yamamoto [16, 18]. Also see Imanuvilov and Yamamoto [19] for uniqueness results for elliptic systems. In Guillarmou and Tzou [13], the result of [15] was extended on Riemannian surfaces. In particular, for uniqueness in determining a two-dimensional potential with full data: $\Lambda_{q,\partial\Omega}$, we refer to Blasten [4], Blasten, Imanuvilov and Yamamoto [5], Bukhgeim [8] and, Sun and Uhlmann [26], and for systems in Albin, Guillarmou, Tzou and Uhlmann [1] and Novikov and Santacesaria [24]. For the case of full data, in [4] and [20], it was shown that $\Lambda_{q,\partial\Omega}$ uniquely determines q in the class piecewise $W_p^1(\Omega)$ with $p > 2$ and $C^\alpha(\overline{\Omega})$, $\alpha > 0$, respectively. As for the related problem of recovery of the conductivity in EIT, Astala and Päivärinta [2] proved uniqueness for conductivities in $L^\infty(\Omega)$, improving the results of Nachman [23] and Brown and Uhlmann [7]. Moreover for the case of dimensions $n \geq 3$ with the full data Sylvester and Uhlmann [27] proved the uniqueness of recovery of a conductivity in $C^2(\overline{\Omega})$, and later the regularity assumption was improved (see e.g. Brown and Torres [6], Päivärinta, Panchenko and Uhlmann [25] and Haberman and Tataru [14]). The case when voltages are applied and current is measured on different subsets was studied in dimensions greater than two in Bukhgeim and Uhlmann [9], Kenig, Sjöstrand and Uhlmann [22] and in Imanuvilov, Uhlmann and Yamamoto [17] for the two-dimensional case. See also the reviews by Imanuvilov and Yamamoto [21], and Uhlmann [28].

Our main result is as follows

Theorem 1. *Let $q_1, q_2 \in W_p^1(\Omega)$ for some $p > 2$. If $D(N_{q_1,\widetilde{\Gamma}}) \subset D(N_{q_2,\widetilde{\Gamma}})$ and $N_{q_1,\widetilde{\Gamma}}(f) = N_{q_2,\widetilde{\Gamma}}(f)$ for each f from $D(N_{q_1,\widetilde{\Gamma}})$, then $q_1 = q_2$ in Ω.*

Notice that Theorem 1 does not assume that Ω is simply connected. An interesting inverse problem is whether one can determine the potential in a domain with holes by measuring $N_{q,\widetilde{\Gamma}}$ only on some open set $\widetilde{\Gamma}$ in the outer subboundary.

Let Ω, G be bounded domains in $\mathbb{R}^2$ with smooth boundaries such that $\overline{G} \subset \Omega$. Let $\widetilde{\Gamma} \subset \partial\Omega$ be an open set and $q \in W_p^1(\Omega \setminus \overline{G})$ with some $p > 2$. Consider the following Neumann-to-Dirichlet map:

$$\widetilde{N}_{q,\widetilde{\Gamma}} : f \to u|_{\widetilde{\Gamma}},$$

where

$$u \in H^1(\Omega \setminus \overline{G}), \quad (\Delta + q)u = 0 \text{ in } \Omega \setminus \overline{G}, \quad \frac{\partial u}{\partial \nu}|_{\partial G \cup (\partial\Omega \setminus \widetilde{\Gamma})} = 0, \quad \frac{\partial u}{\partial \nu}|_{\widetilde{\Gamma}} = f.$$

Then we can directly derive the following from Theorem 1.

Corollary 2. *Let* $q_1, q_2 \in W^1_p(\Omega \setminus \overline{G})$ *with some* $p > 2$*. If* $D(\widetilde{N}_{q_1,\widetilde{\Gamma}}) \subset D(\widetilde{N}_{q_2,\widetilde{\Gamma}})$ *and* $\widetilde{N}_{q_1,\widetilde{\Gamma}}(f) = \widetilde{N}_{q_2,\widetilde{\Gamma}}(f)$ *for each* f *from* $D(\widetilde{N}_{q_1,\widetilde{\Gamma}})$*, then* $q_1 = q_2$ *in* $\Omega \setminus \overline{G}$*.*

For the case of EIT, if the conductivities are known on $\widetilde{\Gamma}$, then we can apply our theorem to prove uniqueness of the determination of conductivities in $W^3_p(\Omega), p > 2$ from the Neumann-to-Dirichlet map.

The remainder of the paper is devoted to the proof of Theorem 1. The main technique is the construction of complex geometrical optics solutions whose Neumann data vanish on the complement of $\widetilde{\Gamma}$.

Throughout the article, we use the following notations.

Notations. We set $\Gamma_0 = \partial\Omega \setminus \overline{\widetilde{\Gamma}}$, $i = \sqrt{-1}$, $x_1, x_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$ and $\xi = (\xi_1, \xi_2)$ with $\zeta = \xi_1 + i\xi_2$. $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$, $D = \left(\frac{1}{i}\partial_{x_1}, \frac{1}{i}\partial_{x_2}\right)$, $\partial_\zeta = \frac{1}{2}(\partial_{\xi_1} - i\partial_{\xi_2})$, $\partial_{\bar{\zeta}} = \frac{1}{2}(\partial_{\xi_1} + i\partial_{\xi_2})$. Denote by $B(x, \delta)$ a ball centered at x of radius δ. For a normed space X, by $o_X(\frac{1}{\tau^\kappa})$ we denote a function $f(\tau, \cdot)$ such that $\|f(\tau, \cdot)\|_X = o(\frac{1}{\tau^\kappa})$ as $|\tau| \to +\infty$. The tangential derivative on the boundary is given by $\partial_{\vec{\tau}} = \nu_2 \frac{\partial}{\partial x_1} - \nu_1 \frac{\partial}{\partial x_2}$, where $\nu = (\nu_1, \nu_2)$ is the unit outer normal to $\partial\Omega$. The operators ∂_z^{-1} and $\partial_{\bar{z}}^{-1}$ are given by

$$\partial_{\bar{z}}^{-1} g = -\frac{1}{\pi} \int_\Omega \frac{g(\zeta, \bar{\zeta})}{\zeta - z} d\xi_2 d\xi_1, \quad \partial_z^{-1} g = \overline{\partial_{\bar{z}}^{-1} \bar{g}}.$$

We call $b(x)$ antiholomorphic if $(\overline{b(x)})$ is holomorphic. In the Sobolev space $H^1(\Omega)$ we introduce the following norm

$$\|u\|_{H^{1,\tau}(\Omega)} = (\|u\|^2_{H^1(\Omega)} + |\tau|^2 \|u\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

2. Proof of Theorem 1

Let $\Phi = \varphi + i\psi$ be a holomorphic function in Ω such that φ, ψ are real-valued and

$$\Phi \in C^2(\overline{\Omega}), \quad \operatorname{Im} \Phi|_{\Gamma_0^*} = 0, \quad \Gamma_0 \subset\subset \Gamma_0^* \quad \Gamma_0^* \neq \partial\Omega, \tag{4}$$

where Γ_0^* is some open set in $\partial\Omega$. Denote by $\mathcal{H}$ the set of the critical points of the function Φ. Assume that

$$\mathcal{H} \neq \emptyset, \quad \partial_z^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}, \quad \mathcal{H} \cap \overline{\widetilde{\Gamma}} = \emptyset \tag{5}$$

and

$$\int_{\mathcal{J}} 1 d\sigma = 0, \quad \mathcal{J} = \{x;\, \partial_{\vec{\tau}}\psi(x) = 0, x \in \partial\Omega \setminus \Gamma_0^*\}. \tag{6}$$

Let Ω_1 be a bounded domain in $\mathbb{R}^2$ such that $\Omega \subset\subset \Omega_1$ and $\mathcal{C}$ be some smooth complex-valued function in Ω such that

$$2\frac{\partial \mathcal{C}}{\partial z} = C_1(x) + iC_2(x) \quad \text{in } \Omega_1, \tag{7}$$

where C_1, C_2 are smooth real-valued functions in Ω such that

$$\frac{\partial C_1}{\partial x_1} + \frac{\partial C_2}{\partial x_2} = 1 \quad \text{in} \quad \Omega_1. \tag{8}$$

The following proposition is proved as Proposition 2.5 in [18].

Proposition 1. *Suppose that $q \in L^\infty(\Omega)$, the function Φ satisfies (4), (5), and the function $\mathcal{C}$ satisfies (7), (8) and $\widetilde{v} \in H^2(\Omega)$. Then there exist τ_0 and $C(N)$ independent of $\widetilde{v}$ and τ such that*

$$\begin{aligned} &\frac{N}{2}\|2\partial_{\bar z}\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \tau\|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{H^1(\Omega)} \\ &\quad + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} \\ &\le \|L_q(x,D)\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + C(N)\tau\|(\widetilde{v}e^{\tau\varphi+N\mathcal{C}}, \frac{\partial\widetilde{v}}{\partial\nu}e^{\tau\varphi+N\mathcal{C}})\|^2_{H^{1,\tau}(\partial\Omega)\times L^2(\partial\Omega)} \end{aligned} \tag{9}$$

for all $\tau > \tau_0(N)$ and all positive $N \ge 1$.

Let $\widetilde{v} \in H^2(\Omega)$ satisfy

$$\frac{\partial\widetilde{v}}{\partial\nu}|_{\Gamma_0^*} = 0.$$

Using Proposition 1, we can show the following.

Proposition 2. *Suppose that Φ satisfies (4), (5) and $q \in L^\infty(\Omega)$. Then there exist τ_0 and C independent of $\widetilde{v}$ and τ such that*

$$\begin{aligned} &\tau\|\widetilde{v}e^{\tau\varphi}\|^2_{L^2(\Omega)} + \|\widetilde{v}e^{\tau\varphi}\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}e^{\tau\varphi}\|^2_{L^2(\Omega)} \\ &\le C\left(\|L_q(x,D)\widetilde{v}e^{\tau\varphi}\|^2_{L^2(\Omega)} + \tau\|(\widetilde{v}e^{\tau\varphi}, \frac{\partial\widetilde{v}e^{\tau\varphi}}{\partial\nu})\|^2_{H^{1,\tau}(\widetilde{\Gamma})\times L^2(\widetilde{\Gamma})}\right) \end{aligned} \tag{10}$$

for all $\tau > \tau_0$ and for all $\widetilde{v} \in H^2(\Omega)$ and $\frac{\partial\widetilde{v}}{\partial\nu}|_{\Gamma_0^*} = 0$.

Proof. Let $\{e_j\}_{j=1}^M$ be a partition of unity such that $e_j \in C_0^\infty(B(x_j,\delta))$ where x_j are some points in Ω,

$$\sum_{j=1}^M e_j(x) = 1 \quad \text{on } \Omega, \quad \frac{\partial e_j}{\partial \nu}|_{\Gamma_0^*} = 0 \quad \forall j \in \{1,\dots,M\},$$

and let δ be a small positive number such that $B(x_j,\delta)\cap\Gamma_0 \neq \emptyset$ implies $B(x_j,\delta)\cap\partial\Omega \subset \Gamma_0^*$. Denote $w_j = e_j\widetilde{v}$. Let $\operatorname{supp} w_j \cap (\partial\Omega\setminus\widetilde{\Gamma}) = \emptyset$. Then Proposition 1 implies that there exists τ_0 such that for all $\tau \ge \tau_0$

$$\begin{aligned}
&\frac{N}{2}\|2\partial_{\bar z} w_j e^{\tau\varphi} e^{N\mathcal{C}}\|^2_{L^2(\Omega)} + \tau\|w_j e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|w_j e^{\tau\varphi+N\mathcal{C}}\|^2_{H^1(\Omega)} \\
&\quad + \tau^2\||\frac{\partial\Phi}{\partial z}|w_j e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} \\
&\le C\|L_q(x,D)w_j e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + C(N)\|(w_j e^{\tau\varphi}, \frac{\partial w_j e^{\tau\varphi}}{\partial\nu})\|^2_{H^{1,\tau}(\widetilde{\Gamma})\times L^2(\widetilde{\Gamma})}.
\end{aligned} \tag{11}$$

Next let $\operatorname{supp} w_j \cap (\partial\Omega\setminus\widetilde{\Gamma}) \neq \emptyset$. We cannot apply directly the Carleman estimate (10) in this case, since the function w_j may not satisfy the zero Dirichlet boundary condition. To overcame this difficulty we construct an extension. Without loss of generality, using if necessary a conformal transformation, we can assume that $\operatorname{supp} w_j \cap \Omega \subset \{x_2 > 0\}$ and $\operatorname{supp} w_j \cap \partial\Omega \subset \{x_2 = 0\}$. Then using the extension $w_j(x_1,x_2) = w_j(x_1,-x_2)$, $q(x_1,x_2) = q(x_1,-x_2)$ and $\varphi(x_1,x_2) = \varphi(x_1,-x_2)$, we apply Proposition 1 to the operator $L_q(x,D)$ in $\mathcal{O} = \operatorname{supp} e_j \cup \{x|(x_1,-x_2) \in \operatorname{supp} e_j\}$. We have the same estimate (11). Therefore for all $\tau \ge \tau_0$

$$\begin{aligned}
\|\widetilde{v}\|_*^2 &:= \tau\|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{H^1(\Omega)} + \tau^2\||\frac{\partial\Phi}{\partial z}|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} \\
&\le \sum_{j=1}^M \|\widetilde{v}e_j\|_*^2 \\
&\le C\sum_{j=1}^M \|L_q(x,D)w_j e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + C(N)\|(w_j e^{\tau\varphi}, \frac{\partial w_j e^{\tau\varphi}}{\partial\nu})\|^2_{H^{1,\tau}(\widetilde{\Gamma})\times L^2(\widetilde{\Gamma})} \\
&\le C\sum_{j=1}^M (\|\Delta e_j \widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|2\partial_z e_j \partial_{\bar z}\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} \\
&\quad - N\|\partial_{\bar z}(\widetilde{v}e_j)e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|L_q(x,D)\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)}) \\
&\quad + C(N)\|(\widetilde{v}e^{\tau\varphi}, \frac{\partial\widetilde{v}e^{\tau\varphi}}{\partial\nu})\|^2_{H^{1,\tau}(\widetilde{\Gamma})\times L^2(\widetilde{\Gamma})}.
\end{aligned} \tag{12}$$

Fixing the parameter N sufficiently large, we obtain from (12)

$$\|\widetilde{v}\|_*^2 \le C(N)\Big(\|\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|L_q(x,D)\widetilde{v}e^{\tau\varphi+N\mathcal{C}}\|^2_{L^2(\Omega)} + \|(\widetilde{v}e^{\tau\varphi}, \frac{\partial \widetilde{v}e^{\tau\varphi}}{\partial\nu})\|^2_{H^{1,\tau}(\widetilde{\Gamma})\times L^2(\widetilde{\Gamma})}\Big). \tag{13}$$

The first term on the right-hand side of (13) can be absorbed into the left-hand side for all sufficiently large τ. Since N and $\mathcal{C}$ are independent of τ, the proof of the proposition is finished. □

The Carleman estimate (10) implies the existence of solutions to the following boundary value problem.

Proposition 3. *There exists a constant τ_0 such that for $|\tau| \ge \tau_0$ and any $f \in L^2(\Omega)$, $r \in H^{\frac{1}{2}}(\Gamma_0^*)$, there exists a solution to the boundary value problem*

$$L_q(x,D)u = fe^{\tau\varphi} \quad in\ \Omega, \quad \frac{\partial u}{\partial\nu}|_{\Gamma_0} = re^{\tau\varphi} \tag{14}$$

such that

$$\|u\|_{H^{1,\tau}(\Omega)}/\sqrt{|\tau|} \le C(\|f\|_{L^2(\Omega)} + |\tau|^{\frac{1}{4}}\|r\|_{L^2(\Gamma_0)} + \|r\|_{H^{\frac{1}{2}}(\Gamma_0^*)}). \tag{15}$$

The constant C is independent of τ.

The proof of this proposition uses standard duality arguments, see e.g. [15].

We define two other operators:

$$\mathcal{R}_\tau g = \frac{1}{2}e^{\tau(\Phi-\overline{\Phi})}\partial_{\overline{z}}^{-1}(ge^{\tau(\overline{\Phi}-\Phi)}),\ \widetilde{\mathcal{R}}_\tau g = \frac{1}{2}e^{\tau(\overline{\Phi}-\Phi)}\partial_z^{-1}(ge^{\tau(\Phi-\overline{\Phi})}). \tag{16}$$

Observe that

$$2\frac{\partial}{\partial z}(e^{\tau\Phi}\widetilde{\mathcal{R}}_\tau g) = ge^{\tau\Phi}, \quad 2\frac{\partial}{\partial \overline{z}}(e^{\tau\overline{\Phi}}\mathcal{R}_\tau g) = ge^{\tau\overline{\Phi}} \quad \forall g \in L^2(\Omega). \tag{17}$$

Let $a \in C^6(\overline{\Omega})$ be some holomorphic function in Ω such that

$$\operatorname{Im} a|_{\Gamma_0^*} = 0, \quad \lim_{z\to\hat{z}} a(z)/|z-\hat{z}|^{100} = 0, \quad \forall \hat{z} \in \mathcal{H}\cap\Gamma_0^*. \tag{18}$$

Moreover, for some $\widetilde{x} \in \mathcal{H}$, we assume that

$$a(\widetilde{x}) \ne 0 \quad \text{and} \quad a(x) = 0, \quad \forall x \in \mathcal{H}\setminus\{\widetilde{x}\}. \tag{19}$$

The existence of such a function is proved in Proposition 9 of [19]. Let polynomials $M_1(z)$ and $M_3(\overline{z})$ satisfy

$$(\partial_{\bar z}^{-1}q_1 - M_1)(\widetilde{x}) = 0, \qquad (\partial_{z}^{-1}q_1 - M_3)(\widetilde{x}) = 0. \tag{20}$$

The holomorphic function a_1 and the antiholomorphic function b_1 are defined by formulae $a_1(z) = a_{1,1}(z) + a_{1,2}(z) + a_{1,3}(z)$ and $b_1(\bar z) = b_{1,1}(\bar z) + b_{1,2}(\bar z) + b_{1,3}(\bar z)$ where $a_{1,1}, b_{1,1} \in C^1(\overline{\Omega})$ and

$$\begin{aligned} &i\frac{\partial\psi}{\partial\nu}a_{1,1}(z) - i\frac{\partial\psi}{\partial\nu}b_{1,1}(\bar z) \\ &= -\frac{\partial(a+\bar a)}{\partial\nu} + i\frac{\partial\psi}{\partial\nu}\frac{a(\partial_{\bar z}^{-1}q_1 - M_1)}{4\partial_z\Phi} - i\frac{\partial\psi}{\partial\nu}\frac{\bar a(\partial_{z}^{-1}q_1 - M_3)}{4\partial_{\bar z}\overline{\Phi}} \quad \text{on } \Gamma_0^* \end{aligned} \tag{21}$$

and $a_{1,2}(z,\tau), b_{1,2}(\bar z,\tau) \in C^1(\overline{\Omega})$ for each τ are holomorphic and antiholomorphic functions such that

$$b_{1,2}(\bar z,\tau) = -\frac{1}{8\pi}\int_{\partial\Omega}\frac{(\nu_1 - i\nu_2)a(\partial_{\bar\zeta}^{-1}q_1 - M_1)e^{\tau(\Phi-\overline{\Phi})}}{(\bar\zeta - \bar z)\partial_\zeta\Phi}d\sigma$$

and

$$a_{1,2}(z,\tau) = -\frac{1}{8\pi}\int_{\partial\Omega}\frac{(\nu_1 + i\nu_2)\bar a(\partial_{\zeta}^{-1}q_1 - M_3)e^{\tau(\overline{\Phi}-\Phi)}}{(\zeta - z)\partial_{\bar\zeta}\overline{\Phi}}d\sigma.$$

Here the denominators of the integrands vanish in $\mathcal{H}\cap\Gamma_0^*$, but thanks to the second condition in (18) integrability is guaranteed. We represent the functions $a_{1,2}(z,\tau), b_{1,2}(\bar z,\tau)$ in the form

$$a_{1,2}(z,\tau) = a_{1,2,1}(z) + a_{1,2,2}(z,\tau), \quad b_{1,2}(\bar z,\tau) = b_{1,2,1}(\bar z) + b_{1,2,2}(\bar z,\tau),$$

where

$$b_{1,2,1}(\bar z) = -\frac{1}{8\pi}\int_{\Gamma_0^*}\frac{(\nu_1 - i\nu_2)a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{(\bar\zeta - \bar z)\partial_\zeta\Phi}d\sigma,$$

$$a_{1,2,1}(z) = -\frac{1}{8\pi}\int_{\Gamma_0^*}\frac{(\nu_1 + i\nu_2)\bar a(\partial_{\zeta}^{-1}q_1 - M_3)}{(\zeta - z)\partial_{\bar\zeta}\overline{\Phi}}d\sigma.$$

By (18), the functions $b_{1,2,1}$, $a_{1,2,1}$ belong to $C^1(\overline{\Omega})$. By (6) we have

$$\|b_{1,2,1}(\cdot,\tau)\|_{L^2(\Omega)} + \|a_{1,2,1}(\cdot,\tau)\|_{L^2(\Omega)} \to 0 \quad \text{as } \tau \to +\infty.$$

Finally $a_{1,3}(z,\tau), b_{1,3}(\bar z,\tau) \in H^1(\Omega)$ for each τ are holomorphic and antiholomorphic functions respectively such that

$$\begin{aligned}
&i\frac{\partial\psi}{\partial\nu}a_{1,3}(z,\tau) - i\frac{\partial\psi}{\partial\nu}b_{1,3}(\bar z,\tau)\\
&= \frac{i}{2\pi}\frac{\partial\psi}{\partial\nu}\int\limits_\Omega \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{(\bar\zeta-\bar z)}d\xi_2 d\xi_1\\
&\quad - \frac{i}{2\pi}\frac{\partial\psi}{\partial\nu}\int\limits_\Omega \partial_{\bar\zeta}\left(\frac{\bar a(\partial_{\bar\zeta}^{-1}q_1 - M_3)}{\partial_{\bar\zeta}\bar\Phi}\right)\frac{e^{\tau(\bar\Phi-\Phi)}}{(\zeta-z)}d\xi_2 d\xi_1 \quad \text{on } \Gamma_0^* \qquad (22)
\end{aligned}$$

and

$$\|a_{1,3}(\cdot,\tau)\|_{L^2(\Omega)} + \|b_{1,3}(\cdot,\tau)\|_{L^2(\Omega)} = o(1) \quad \text{as } \tau\to+\infty. \qquad (23)$$

The inequality (23) follows from the asymptotic formula

$$\begin{aligned}
&\left\|\int\limits_\Omega \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1\right\|_{H^{\frac12}(\Gamma_0^*)}\\
&+\left\|\int\limits_\Omega \partial_{\bar\zeta}\left(\frac{\bar a(\partial_{\bar\zeta}^{-1}q_1 - M_3)}{\partial_{\bar\zeta}\bar\Phi}\right)\frac{e^{\tau(\bar\Phi-\Phi)}}{\zeta-z}d\xi_2 d\xi_1\right\|_{H^{\frac12}(\Gamma_0^*)} = o(1) \quad \text{as } \tau\to+\infty. \qquad (24)
\end{aligned}$$

In order to prove (24) consider the function $e\in C_0^\infty(\Omega)$ such that $e\equiv 1$ in some neighborhood of the set $\mathcal{H}\setminus\Gamma_0^*$. The family of functions $\int_\Omega e\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1 \in C^\infty(\partial\Omega)$, are uniformly bounded in τ in $C^2(\partial\Omega)$ and by Proposition 2.4 of [15] this function converges pointwisely to zero. Therefore

$$\left\|\int\limits_\Omega e\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1\right\|_{H^1(\partial\Omega)} = o(1) \quad \text{as } \tau\to+\infty. \qquad (25)$$

Integrating by parts we obtain

$$\begin{aligned}
&\int\limits_\Omega (1-e)\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1\\
&= \frac{(1-e)}{\partial_z\Phi}\partial_z\left(\frac{a(\partial_{\bar z}^{-1}q_1 - M_1)}{\tau\partial_z\Phi}\right)e^{\tau(\Phi-\bar\Phi)}\\
&\quad - \frac{1}{\tau}\int\limits_\Omega \partial_\zeta\left(\frac{(1-e)}{\partial_\zeta\Phi}\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1.
\end{aligned}$$

Thanks to (4) and (18), we have

$$\left\|\frac{1-e}{\partial_z\Phi}\partial_z\left(\frac{a(\partial_{\bar z}^{-1}q_1-M_1)}{\tau\partial_z\Phi}\right)e^{\tau(\Phi-\overline{\Phi})}\right\|_{H^{\frac12}(\Gamma_0^*)}=o(1)\quad\text{as }\tau\to+\infty. \tag{26}$$

The functions $\partial_\zeta\left(\frac{1-e}{\partial_\zeta\Phi}\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\right)e^{\tau(\Phi-\overline{\Phi})}$ are bounded in $L^p(\Omega)$ uniformly in τ. Therefore by Proposition 2.2 of [15], the functions $\int_\Omega\partial_\zeta\left(\frac{1-e}{\partial_\zeta\Phi}\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\times\frac{e^{\tau(\Phi-\overline{\Phi})}}{\bar\zeta-\bar z}\right)d\xi_2d\xi_1$ are uniformly bounded in $W_p^1(\Omega)$. The trace theorem yields

$$\left\|\frac{1}{\tau}\int_\Omega\partial_\zeta\left(\frac{1-e}{\partial_\zeta\Phi}\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\overline{\Phi})}}{\bar\zeta-\bar z}\right)d\xi_2d\xi_1\right\|_{H^{\frac12}(\Gamma_0^*)}=o(1)$$
$$\text{as }\tau\to+\infty. \tag{27}$$

By (25)–(27) we obtain (24).

We note that by (18) the function $\frac{a}{\partial_z\Phi}\in C^2(\partial\Omega)$. We define the function U_1 by the formula

$$\begin{aligned}U_1(x)&=e^{\tau\Phi}(a+a_1/\tau)+e^{\tau\overline{\Phi}}(\bar a+b_1/\tau)-\frac12e^{\tau\Phi}\widetilde{\mathcal R}_\tau\{a(\partial_{\bar z}^{-1}q_1-M_1)\}\\&\quad-\frac12e^{\tau\overline{\Phi}}\mathcal R_\tau\{\bar a(\partial_z^{-1}q_1-M_3)\}.\end{aligned} \tag{28}$$

Integrating by parts, we obtain the following:

$$\begin{aligned}&e^{\tau\Phi}\widetilde{\mathcal R}_\tau\{a(\partial_{\bar z}^{-1}q_1-M_1)\}\\&=\frac1\tau\Bigg(2b_{1,2}e^{\tau\overline{\Phi}}+\frac{e^{\tau\Phi}a(\partial_{\bar z}^{-1}q_1-M_1)}{2\partial_z\Phi}\\&\quad+\frac{e^{\tau\overline{\Phi}}}{2\pi}\int_\Omega\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\overline{\Phi})}}{\bar\zeta-\bar z}d\xi_2d\xi_1\Bigg)\end{aligned} \tag{29}$$

and

$$\begin{aligned}&e^{\tau\overline{\Phi}}\mathcal R_\tau\{\bar a(\partial_z^{-1}q_1-M_3)\}\\&=\frac1\tau\Bigg(2a_{1,2}e^{\tau\Phi}+\frac{e^{\tau\overline{\Phi}}\bar a(\partial_z^{-1}q_1-M_3)}{2\partial_{\bar z}\overline{\Phi}}\\&\quad+\frac{e^{\tau\Phi}}{2\pi}\int_\Omega\partial_{\bar\zeta}\left(\frac{\bar a(\partial_{\zeta}^{-1}q_1-M_3)}{\partial_{\bar\zeta}\overline{\Phi}}\right)\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\zeta-z}d\xi_2d\xi_1\Bigg).\end{aligned} \tag{30}$$

We claim that

$$\left\|\frac{e^{-i\tau\psi}}{2\pi}\int_\Omega \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1\right\|_{L^2(\Omega)} + \left\|\frac{e^{i\tau\psi}}{2\pi}\int_{\partial\Omega} \partial_{\bar\zeta}\left(\frac{\bar a(\partial_{\bar\zeta}^{-1}q_1 - M_3)}{\partial_{\bar\zeta}\bar\Phi}\right)\frac{e^{\tau(\bar\Phi-\Phi)}}{\zeta - z}d\xi_2 d\xi_1\right\|_{L^2(\Omega)} \to 0 \quad \text{as } \tau\to+\infty. \tag{31}$$

We prove the asymptotic formula (31) for the first term. The proof of the asymptotic for the second term is the same. Denote $r_\tau(\xi) = \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)e^{\tau(\Phi-\bar\Phi)}$. By (5), (19) and (20), the family of these functions is bounded in $L^p(\Omega)$ for any $p < 2$. Hence by Proposition 2.2 of [15] there exists a constant C independent of τ such that

$$\left\|\frac{e^{-i\tau\psi}}{2\pi}\int_\Omega \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1\right\|_{L^4(\Omega)} \le C. \tag{32}$$

By (5), (19) and (20), for any $z \ne \widetilde{x}_1 + i\widetilde{x}_2$, the function $r_\tau(\xi)/(\bar\zeta - \bar z)$ belongs to $L^1(\Omega)$. Therefore by Proposition 2.4 of [15], we have

$$\frac{e^{-i\tau\psi}}{2\pi}\int_\Omega \partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1 - M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\bar\Phi)}}{\bar\zeta-\bar z}d\xi_2 d\xi_1 \to 0 \quad \text{a.e. in} \quad \Omega. \tag{33}$$

From (32), (33) and Egorov's theorem, the asymptotic for the first term in (31) follows immediately.

We set

$$\begin{aligned} g_\tau = q_1(e^{i\tau\psi}a_1/\tau + e^{-i\tau\psi}b_1/\tau - \frac{e^{i\tau\psi}}{2}\widetilde{\mathcal{R}}_\tau\{a(\partial_{\bar z}^{-1}q_1 - M_1)\} \\ - \frac{e^{-i\tau\psi}}{2}\mathcal{R}_\tau\{\bar a(\partial_z^{-1}q_1 - M_3)\}). \end{aligned}$$

By (29)–(31) we have

$$\|g_\tau\|_{L^2(\Omega)} = O(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{34}$$

Short computations give

$$L_1(x,D)U_1 = e^{\tau\varphi}g_\tau \quad \text{in } \Omega, \qquad \frac{\partial U_1}{\partial\nu}|_{\Gamma_0} = e^{\tau\varphi}O_{H^{\frac12}(\overline{\Gamma_0^*})}(\frac{1}{\tau}) \quad \text{as } \tau\to+\infty. \tag{35}$$

Indeed, the first equation in (35) follows from (28), (19) and the factorization of the Laplace operator in the form $\Delta = 4\partial_{\bar z}\partial_z$. In order to prove the second equation in (35) we set $\frac{\partial U_1}{\partial\nu} = I_1 + I_2$ where

$$
\begin{aligned}
I_1 &= \frac{\partial}{\partial\nu}((a+a_{1,1}/\tau)e^{\tau\Phi}+(\bar a+b_{1,1}/\tau)e^{\tau\overline{\Phi}})\\
&= e^{\tau\varphi}\left(i\tau\frac{\partial\psi}{\partial\nu}(a+a_{1,1}/\tau)-i\tau\frac{\partial\psi}{\partial\nu}(\bar a+b_{1,1}/\tau)+\frac{\partial}{\partial\nu}(a+a_{1,1}/\tau)+\frac{\partial}{\partial\nu}(\bar a+b_{1,1}/\tau)\right)\\
&= \left(i\frac{\partial\psi}{\partial\nu}\frac{a(\partial_{\bar z}^{-1}q_1-M_1)}{4\partial_z\Phi}-i\frac{\partial\psi}{\partial\nu}\frac{\bar a(\partial_z^{-1}q_1-M_3)}{4\partial_{\bar z}\overline{\Phi}}\right)e^{\tau\varphi}+e^{\tau\varphi}O_{C^1(\overline{\Gamma_0^*})}(\frac{1}{\tau}).
\end{aligned}
\tag{36}
$$

In order to obtain the last equality, we used (18) and (21). Then

$$
\begin{aligned}
I_2 &= \frac{\partial}{\partial\nu}((a_{1,2}+a_{1,3})e^{\tau\Phi}+(b_{1,2}+b_{1,3})e^{\tau\overline{\Phi}})-\frac{1}{2}\frac{\partial}{\partial\nu}(e^{\tau\Phi}\widetilde{\mathcal{R}}_\tau\{a(\partial_{\bar z}^{-1}q_1-M_1)\}\\
&\quad+e^{\tau\overline{\Phi}}\mathcal{R}_\tau\{\bar a(\partial_z^{-1}q_1-M_3)\})\\
&= -\frac{1}{2}\frac{\partial}{\partial\nu}\left(\frac{e^{\tau\Phi}a(\partial_{\bar z}^{-1}q_1-M_1)}{2\partial_z\Phi}+\frac{e^{\tau\overline{\Phi}}}{2\pi}\int_\Omega\partial_\zeta\left(\frac{a(\partial_{\bar\zeta}^{-1}q_1-M_1)}{\partial_\zeta\Phi}\right)\frac{e^{\tau(\Phi-\overline{\Phi})}}{\bar\zeta-\bar z}d\xi_2d\xi_1\right.\\
&\quad\left.+\frac{e^{\tau\overline{\Phi}}\bar a(\partial_z^{-1}q_1-M_3)}{2\partial_{\bar z}\overline{\Phi}}+\frac{e^{\tau\Phi}}{2\pi}\int_\Omega\partial_{\bar\zeta}\left(\frac{\bar a(\partial_{\zeta}^{-1}q_1-M_3)}{\partial_{\bar\zeta}\overline{\Phi}}\right)\frac{e^{\tau(\overline{\Phi}-\Phi)}}{\zeta-z}d\xi_2d\xi_1\right)\\
&= -i\frac{\partial\psi}{\partial\nu}\left(\frac{a(\partial_{\bar z}^{-1}q_1-M_1)}{4\partial_z\Phi}-\frac{e^{\tau\overline{\Phi}}\bar a(\partial_z^{-1}q_1-M_3)}{4\partial_{\bar z}\overline{\Phi}}\right)+O_{H^{\frac12}(\Gamma_0^*)}(\frac{1}{\tau}).
\end{aligned}
\tag{37}
$$

From (36) and (37), we obtain the second equation in (35).

Finally we construct the last term of the complex geometric optics solution $e^{\tau\varphi}w_\tau$. Consider the boundary value problem

$$
L_{q_1}(x,D)(w_\tau e^{\tau\varphi})=-g_\tau e^{\tau\varphi}\quad\text{in }\Omega,\quad \frac{\partial(w_\tau e^{\tau\varphi})}{\partial\nu}|_{\Gamma_0}=-\frac{\partial U_1}{\partial\nu}.
\tag{38}
$$

By (34) and Proposition 3, there exists a solution to problem (38) such that

$$
\|w_\tau\|_{L^2(\Omega)}=o(\frac{1}{\tau})\quad\text{as }\tau\to+\infty.
\tag{39}
$$

Finally we set

$$
u_1=U_1+e^{\tau\varphi}w_\tau.
\tag{40}
$$

By (39), (40), (23) and (28)–(30), we can represent the complex geometric optics solution u_1 in the form

$$
\begin{aligned}
u_1(x) &= e^{\tau\Phi}(a+(a_{1,1}+a_{1,2,1})/\tau)+e^{\tau\overline{\Phi}}(\bar a+(b_{1,1}+b_{1,2,1})/\tau)\\
&\quad-\left(e^{\tau\Phi}\frac{a(\partial_{\bar z}^{-1}q_1-M_1)}{4\tau\partial_z\Phi}+e^{\tau\overline{\Phi}}\frac{\bar a(\partial_z^{-1}q_1-M_3)}{4\tau\overline{\partial_z\Phi}}\right)+e^{\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau})\\
&\quad\text{as }\tau\to+\infty.
\end{aligned}
\tag{41}
$$

Since the Cauchy data (2) for the potentials q_1 and q_2 are equal, there exists a solution u_2 to the Schrödinger equation with potential q_2 such that $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$ and $u_1 = u_2$ on $\widetilde{\Gamma}$. Setting $u = u_1 - u_2$, we obtain

$$(\Delta + q_2)u = (q_2 - q_1)u_1 \quad \text{in } \Omega, \quad u|_{\widetilde{\Gamma}} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0. \tag{42}$$

In a similar way to the construction of u_1, we construct a complex geometrical optics solution v for the Schrödinger equation with potential q_2. The construction of v repeats the corresponding steps of the construction of u_1. The only difference is that instead of q_1 and τ, we use q_2 and $-\tau$, respectively. We skip the details of the construction and point out that similarly to (41) it can be represented in the form

$$\begin{aligned} v(x) &= e^{-\tau\Phi}(a + (\widetilde{a}_{1,1} + \widetilde{a}_{1,2,1})/\tau) + e^{-\tau\overline{\Phi}}(\bar{a} + (\widetilde{b}_{1,1} + \widetilde{b}_{1,2,1})/\tau) \\ &\quad + \left(e^{-\tau\Phi}\frac{a(\partial_{\bar{z}}^{-1}q_2 - M_2)}{4\tau\partial_z\Phi} + e^{-\tau\overline{\Phi}}\frac{\bar{a}(\partial_z^{-1}q_2 - M_4)}{4\tau\overline{\partial_z\Phi}}\right) + e^{-\tau\varphi}o_{L^2(\Omega)}(\frac{1}{\tau}) \\ &\text{as } \tau \to +\infty, \quad \frac{\partial v}{\partial \nu}|_{\Gamma_0} = 0, \end{aligned} \tag{43}$$

where $M_2(z)$ and $M_4(\bar{z})$ satisfy

$$(\partial_{\bar{z}}^{-1}q_2 - M_2)(\widetilde{x}) = 0, \qquad (\partial_z^{-1}q_2 - M_4)(\widetilde{x}) = 0.$$

The functions $\widetilde{a}_1(z) = \widetilde{a}_{1,1}(z) + \widetilde{a}_{1,2}(z)$ and $\widetilde{b}_1(z) = \widetilde{b}_{1,1}(z) + \widetilde{b}_{1,2}(z)$ are given by

$$\begin{aligned} &-i\frac{\partial\psi}{\partial\nu}\widetilde{a}_{1,1}(z) + i\frac{\partial\psi}{\partial\nu}\widetilde{b}_{1,1}(\bar{z}) \\ &\quad = -\frac{\partial(a+\bar{a})}{\partial\nu} + i\frac{\partial\psi}{\partial\nu}\frac{a(\partial_{\bar{z}}^{-1}q_2 - M_2)}{4\tau\partial_z\Phi} - i\frac{\partial\psi}{\partial\nu}\frac{\bar{a}(\partial_z^{-1}q_2 - M_4)}{4\tau\overline{\partial_z\Phi}} \quad \text{on } \Gamma_0, \end{aligned}$$

$$\widetilde{a}_{1,1}, \widetilde{b}_{1,1} \in C^1(\overline{\Omega}) \tag{44}$$

and $\widetilde{a}_{1,2,1}(z), \widetilde{b}_{1,2,1}(\bar{z}) \in C^1(\overline{\Omega})$ are holomorphic functions such that

$$\widetilde{b}_{1,2,1}(\bar{z}) = \frac{1}{8\pi}\int\limits_{\Gamma_0^*}\frac{(\nu_1 - i\nu_2)a(\partial_{\bar{\zeta}}^{-1}q_2 - M_2)e^{\tau(\Phi-\overline{\Phi})}}{(\bar{\zeta} - \bar{z})\partial_\zeta\Phi}d\sigma$$

and

$$\widetilde{a}_{1,2,1}(z) = \frac{1}{8\pi}\int\limits_{\Gamma_0^*}\frac{(\nu_1 + i\nu_2)\bar{a}(\partial_\zeta^{-1}q_2 - M_4)e^{\tau(\overline{\Phi}-\Phi)}}{(\zeta - z)\partial_{\bar{\zeta}}\overline{\Phi}}d\sigma.$$

Denote $q = q_1 - q_2$. Taking the scalar product of equation (42) with the function v, we have:

$$\int_\Omega q u_1 v dx = 0. \tag{45}$$

From formulae (41) and (43) in the construction of complex geometrical optics solutions, we have

$$\begin{aligned}
0 &= \int_\Omega q u_1 v dx \\
&= \int_\Omega q(a^2 + \bar{a}^2) dx \\
&\quad + \frac{1}{\tau} \int_\Omega q(a(a_{1,1} + a_{1,2,1} + b_{1,1} + b_{1,2,1}) + \bar{a}(\widetilde{a}_{1,1} + \widetilde{a}_{1,2,1} + \widetilde{b}_{1,1} + \widetilde{b}_{1,2,1}))dx \\
&\quad + \int_\Omega q(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi})dx \\
&\quad + \frac{1}{4\tau} \int_\Omega \left(qa^2 \frac{\partial_{\bar{z}}^{-1} q_2 - M_2}{\partial_z \Phi} + q\bar{a}^2 \frac{\partial_z^{-1} q_2 - M_4}{\overline{\partial_z \Phi}} \right) dx \\
&\quad - \frac{1}{4\tau} \int_\Omega \left(qa^2 \frac{\partial_{\bar{z}}^{-1} q_1 - M_1}{\partial_z \Phi} + q\bar{a}^2 \frac{\partial_z^{-1} q_1 - M_3}{\overline{\partial_z \Phi}} \right) dx \\
&\quad + o(\frac{1}{\tau}) = 0 \quad \text{as } \tau \to +\infty.
\end{aligned} \tag{46}$$

Since the potentials q_j are not necessarily from $C^2(\overline{\Omega})$, we cannot directly use the stationary phase argument (e.g. Evans [12]). Let function $\hat{q} \in C_0^\infty(\Omega)$ satisfy $\hat{q}(\widetilde{x}) = q(\widetilde{x})$. We have

$$\int_\Omega q \,\mathrm{Re}(a\bar{a}e^{2\tau i\psi})dx = \int_\Omega \hat{q} \,\mathrm{Re}(a\bar{a}e^{2\tau i\psi})dx + \int_\Omega (q - \hat{q}) \,\mathrm{Re}(a\bar{a}e^{2\tau i\psi})dx. \tag{47}$$

Using the stationary phase argument and (19), similarly to [15], we obtain

$$\int_\Omega \hat{q}(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi})dx = \frac{2\pi(q|a|^2)(\widetilde{x}) \,\mathrm{Re}\, e^{2\tau i\psi(\widetilde{x})}}{\tau|(\det \psi'')(\widetilde{x})|^{\frac{1}{2}}} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty. \tag{48}$$

For the second integral in (47) we obtain

$$\begin{aligned}
&\int_\Omega (q - \hat{q})(a\bar{a}e^{2\tau i\psi} + a\bar{a}e^{-2\tau i\psi})dx \\
&= \int_\Omega (q - \hat{q}) \left(a\bar{a} \frac{(\nabla\psi, \nabla)e^{2\tau i\psi}}{2\tau i|\nabla\psi|^2} - a\bar{a} \frac{(\nabla\psi, \nabla)e^{-2\tau i\psi}}{2\tau i|\nabla\psi|^2} \right) dx
\end{aligned}$$

$$= \int_{\partial\Omega} q \left(a\bar{a} \frac{(\nabla\psi, \nu) e^{2\tau i \psi}}{2\tau i |\nabla\psi|^2} - a\bar{a} \frac{(\nabla\psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma$$
$$- \frac{1}{2\tau i} \int_{\Omega} \left\{ e^{2\tau i\psi} \operatorname{div} \left((q - \hat{q}) a\bar{a} \frac{\nabla\psi}{|\nabla\psi|^2} \right) - e^{-2\tau i\psi} \operatorname{div} \left((q - \hat{q}) a\bar{a} \frac{\nabla\psi}{|\nabla\psi|^2} \right) \right\} dx. \tag{49}$$

Since $\psi|_{\Gamma_0} = 0$ we have

$$\int_{\partial\Omega} qa\bar{a} \left(\frac{(\nabla\psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla\psi|^2} - \frac{(\nabla\psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma = \int_{\widetilde{\Gamma}} \frac{qa\bar{a}}{2\tau i |\nabla\psi|^2} (\nabla\psi, \nu)(e^{2\tau i\psi} - e^{-2\tau i\psi}) d\sigma.$$

By (4), (6) and Proposition 2.4 in [15] we conclude that

$$\int_{\partial\Omega} qa\bar{a} \left(\frac{(\nabla\psi, \nu) e^{2\tau i\psi}}{2\tau i |\nabla\psi|^2} - \frac{(\nabla\psi, \nu) e^{-2\tau i\psi}}{2\tau i |\nabla\psi|^2} \right) d\sigma = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.$$

The last integral over Ω in formula (49) is $o(\frac{1}{\tau})$ and therefore

$$\int_{\Omega} (q - \hat{q})(a\bar{a} e^{2\tau i\psi} + a\bar{a} e^{-2\tau i\psi}) dx = o(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty. \tag{50}$$

Taking into account that $\psi(\widetilde{x}) \neq 0$ and using (48), (50) we have from (46) that

$$\frac{2\pi (q|a|^2)(\widetilde{x})}{|(\det \psi'')(\widetilde{x})|^{\frac{1}{2}}} = 0. \tag{51}$$

Hence $q(\widetilde{x}) = 0$. In [18] it is proved that there exists a holomorphic function Φ such that (4)–(6) are satisfied and a point $\widetilde{x} \in \mathcal{H}$ can be chosen arbitrarily close to any given point in Ω (see [15]). Hence we have $q \equiv 0$. The proof of the theorem is completed. □

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