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Full and Partial Cloaking in Electromagnetic Scattering

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Abstract

In this paper, we consider two regularized transformation-optics cloaking schemes for electromagnetic (EM) waves. Both schemes are based on the blowup construction with the generating sets being, respectively, a generic curve and a planar subset. We derive sharp asymptotic estimates in assessing the cloaking performances of the two constructions in terms of the regularization parameters and the geometries of the cloaking devices. The first construction yields an approximate full-cloak, whereas the second construction yields an approximate partial-cloak. Moreover, by incorporating properly chosen conducting layers, both cloaking constructions are capable of nearly cloaking arbitrary EM contents. This work complements the existing results in Ammari et al. (SIAM J Appl Math 73:2055–2076, 2013), Bao and Liu (SIAM J Appl Math 74:724–742, 2014), Bao et al. (J Math Pure Appl (9) 101:716–733, 2014) on approximate EM cloaks with the generating set being a singular point, and it also extends Deng et al. (On regularized full- and partial-cloaks in acoustic scat- tering. Preprint, arXiv:1502.01174, 2015), Li et al. (Commun Math Phys, 335:671–712, 2015) on regularized full and partial cloaks for acoustic waves governed by the Helmholtz system to the more challenging EM case governed by the full Maxwell system.

1. Introduction

1.1. Background

This paper is concerned with the invisibility cloaking of electromagnetic (EM) waves via the approach of transformation optics. This method, which is based on the transformation properties of the optical material parameters and the invariance properties of the equations modeling the wave phenomena, was pioneered in [16] for electrostatics. For Maxwell's system, the same transformation optics approach was developed in [34]. In two dimensions, Leonhardt [24] used the conformal mapping to cloak rays in the geometric optics approximation. The obtained cloaking materials are called *metamaterials*. The metamaterials proposed in [17,24,34] for the ideal cloaks are singular, and this has aroused great interest in the literature to deal with the singular structures. In [12,30], the authors proposed considering the finite-energy solutions from singularly weighted Sobolev spaces for the underlying singular PDEs, and both acoustic cloaking and EM cloaking were treated. In [11,13,20,21,27], the authors proposed avoiding the singular structures by incorporating regularization into the cloaking construction, and instead of an ideal invisibility cloak, one considers an approximate/near invisibility cloak. The latter approach has been further investigated in [3,4,28] for the conductivity equation and Helmholtz system, modeling electric impedance tomography (EIT) and acoustic wave scattering, respectively, and in [5–7] for the Maxwell system, modeling the EM wave scattering. For all of the above mentioned work on regularized approximate cloaks, the *generating set* is a singular point, and one always achieves the full-cloak; that is, the invisibility is attainable for detecting waves coming from every possible incident/impinging direction, and observations made at every possible angle. In a recent article [25], the authors proposed studying regularized partial/customized cloaks for acoustic waves; that is, the invisibility is only attainable for limited/customized apertures of incidence and observation. The key idea is to properly choose the generating set for the blowup construction of the cloaking device. In [25], the authors only proved qualitative convergence for the proposed partial/customized cloaking construction, and the corresponding result was further quantified in [9] by the authors of the current article.

In this paper, we shall extend [9,25] on the regularized partial/customized cloaks for acoustic wave scattering to the case of electromagnetic wave scattering. We present two near-cloaking schemes with the generating sets being a generic curve or a planar subset, respectively. It is shown that the first scheme yields an approximate full-cloak, whose invisibility effect is attainable for the whole aperture of incidence and observation angles. The result obtained complements [5–7] on approximate full cloaks. However, the generating set for the blowup construction considered in [5–7] is a singular point, whereas in this study, the generating set is a generic curve. The cloaking material in our full-cloaking scheme is less “singular” than those in [5–7], but at the cost of losing some degree of accuracy on the invisibility approximation. The second scheme would yield an approximate partial-cloak with limited apertures of incidence and observation. We derive sharp asymptotic estimates in assessing the cloaking performances in terms of the regularization parameters and the geometries of the generating sets. The estimates are independent of the EM contents being cloaked, which means that the proposed cloaking schemes are capable of nearly cloaking arbitrary EM objects. Compared to [5–7] on approximate full-cloaks, we need to deal with anisotropic geometries, whereas compared to [9,25] on approximate partial-cloaks for acoustic scattering governed by the Helmholtz system, we need to tackle the more challenging Maxwell system. Finally, we refer the readers to [8,14,15,29,36] for surveys on the theoretical and experimental progress on transformation-optics cloaking in the literature.

1.2. Mathematical Formulation

Consider a homogeneous space with the (normalized) EM medium parameters described by the electric permittivity $\varepsilon_0 = \mathbf{I}_{3\times 3}$ and magnetic permeability $\mu_0 = \mathbf{I}_{3\times 3}$. Here and also in what follows, $\mathbf{I}_{3\times 3}$ denotes the identity matrix in $\mathbb{R}^{3\times 3}$. For notational convenience, we also let $\sigma_0 := 0 \cdot \mathbf{I}_{3\times 3}$ denote the conductivity tensor of the homogeneous background space. We shall consider the invisibility cloaking in the homogeneous space described above. Following the spirit of [5–7], the proposed cloaking device is compactly supported in a bounded domain Ω, and takes a three-layered structure. Let $\Omega_a \Subset \Omega_c \Subset \Omega$ be bounded domains such that Ω_a, $\Omega_c\backslash\overline{\Omega_a}$ and $\Omega\backslash\overline{\Omega_c}$ are connected, and they represent, respectively, the cloaked region, conducting layer and cloaking layer of the proposed cloaking device. Let Γ_0 be a bounded open set in $\mathbb{R}^3$, and it shall be referred to as a generating set in the following. For $\delta \in \mathbb{R}_+$, we let D_δ denote an open neighborhood of Γ_0 such that $D_\delta \to \Gamma_0$ (in the sense of Hausdorff distance) as $\delta \to +0$. D_δ will be referred to as the *virtual domain*, and shall be specified below. Throughout, we assume that there exists a bi-Lipschitz and orientation-preserving mapping $F_\delta : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} F_\delta(D_{\delta/2}) = \Omega_a,\ F_\delta(D_\delta\backslash\overline{D}_{\delta/2}) = \Omega_c\backslash\overline{\Omega}_a,\ F_\delta(\Omega\backslash\overline{D}_\delta) \\ = \Omega\backslash\overline{\Omega}_c \text{ and } F_\delta|_{\mathbb{R}^3\backslash\Omega} = \text{Identity}. \end{aligned} \tag{1.1}$$

Next, we describe the EM medium parameter distributions $\{\mathbb{R}^3; \varepsilon, \mu, \sigma\}$ in the physical space containing the cloaking device, and $\{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ in the virtual space containing the virtual domain. The EM medium parameters are all assumed to be symmetric-positive-definite-matrix valued functions, and they characterize, respectively, the electric permittivity, magnetic permeability and electric conductivity. In what follows, $\{\mathbb{R}^3; \varepsilon, \mu, \sigma\}$ and $\{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ shall be referred to as, respectively, the physical and virtual scattering configurations. Let

$$\{\mathbb{R}^3; \varepsilon, \mu, \sigma\} = \begin{cases} \varepsilon_0, \mu_0, \sigma_0 & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \\ \varepsilon_c^*, \mu_c^*, \sigma_c^* & \text{in } \Omega\backslash\overline{\Omega}_c, \\ \varepsilon_l^*, \mu_l^*, \sigma_l^* & \text{in } \Omega_c\backslash\overline{\Omega}_a, \\ \varepsilon_a^*, \mu_a^*, \sigma_a^* & \text{in } \Omega_a, \end{cases} \tag{1.2}$$

and

$$\{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\} = \begin{cases} \varepsilon_0, \mu_0, \sigma_0 & \text{in } \mathbb{R}^3\backslash\overline{D}_\delta, \\ \varepsilon_l, \mu_l, \sigma_l & \text{in } D_\delta\backslash\overline{D}_{\delta/2}, \\ \varepsilon_a, \mu_a, \sigma_a & \text{in } D_{\delta/2}. \end{cases} \tag{1.3}$$

The virtual and physical scattering configurations are connected by the so-called *push-forward* via the (blowup) transformation F_δ in (1.1). To that end, we next introduce the push-forward of EM mediums. Let m and m_δ, respectively, denote the physical and virtual parameter tensors, where $m = \varepsilon, \mu$ or σ. Define the push-forward $(F_\delta)_* m_\delta$ as

$$m = (F_\delta)_* m_\delta := \left(\frac{1}{\det(DF_\delta)}(DF_\delta)\cdot m_\delta\cdot(DF_\delta)^T\right)\circ F_\delta^{-1}, \tag{1.4}$$

where DF_δ denotes the Jacobian matrix of the transformation F_δ. Throughout the rest of our study, we assume that

$$\{\mathbb{R}^3; \varepsilon, \mu, \sigma\} = (F_\delta)_*\{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\} := \{\mathbb{R}^3; (F_\delta)_*\varepsilon_\delta, (F_\delta)_*\mu_\delta, (F_\delta)_*\sigma_\delta\}. \tag{1.5}$$

Next, we consider the time-harmonic EM wave scattering in the physical space. Let

$$\mathbf{E}^i(\mathbf{x}) := \mathbf{p}e^{i\omega\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{H}^i := \frac{1}{i\omega}(\nabla\times\mathbf{E}^i)(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3, \tag{1.6}$$

where $\mathbf{p}\in\mathbb{R}^3\backslash\{0\}$, $\mathbf{d}\in\mathbb{S}^2$ and $\omega\in\mathbb{R}_+$. $(\mathbf{E}^i, \mathbf{H}^i)$ in (1.6) is called a pair of EM plane waves with $\mathbf{E}^i$ the electric field and $\mathbf{H}^i$ the magnetic field. $\mathbf{p}$ is the polarization tensor, $\mathbf{d}$ is the incident direction and ω is the wavenumber of the plane waves $\mathbf{E}^i$ and $\mathbf{H}^i$. It always holds that

$$\mathbf{p}\perp\mathbf{d}, \quad \text{namely} \quad \mathbf{p}\cdot\mathbf{d} = 0. \tag{1.7}$$

$\mathbf{E}^i$ and $\mathbf{H}^i$ are entire solutions to the following Maxwell equations

$$\begin{cases} \nabla\times\mathbf{E}^i - i\omega\mu_0\mathbf{H}^i = 0 & \text{in } \mathbb{R}^3, \\ \nabla\times\mathbf{H}^i + i\omega\varepsilon_0\mathbf{E}^i = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

The EM scattering in the physical space $\{\mathbb{R}^3; \varepsilon, \mu, \sigma\}$ due to the incident plane waves $(\mathbf{E}^i, \mathbf{H}^i)$ is described by the following Maxwell system

$$\begin{cases} \nabla\times\mathbf{E} - i\omega\mu\mathbf{H} = 0 & \text{in } \mathbb{R}^3, \\ \nabla\times\mathbf{H} + i\omega(\varepsilon + i\frac{\sigma}{\omega})\mathbf{E} = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.8}$$

subject to the Silver-Müller radiation condition:

$$\lim_{\|\mathbf{x}\|\to\infty}\|\mathbf{x}\|\big((\mathbf{H}-\mathbf{H}^i)\times\hat{\mathbf{x}} - (\mathbf{E}-\mathbf{E}^i)\big) = 0, \tag{1.9}$$

where $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|$ for $\mathbf{x}\in\mathbb{R}^3\backslash\{0\}$. We seek solutions $\mathbf{E}, \mathbf{H}\in H_{loc}(\text{curl}; \mathbb{R}^3)$ to (1.8); see [19,22,23,31] for the well-posedness of the scattering system (1.8). Here and also in what follows, we shall often use the spaces

$$H_{loc}(\text{curl}; X) = \{U|_B \in H(\text{curl}; B);\ B \text{ is any bounded subdomain of } X\}$$

and

$$H(\text{curl}; B) = \{U\in(L^2(B))^3;\ \nabla\times U\in(L^2(B))^3\}.$$

It is known that the solution $\mathbf{E}$ to (1.8) admits the following asymptotic expansion as $\|\mathbf{x}\|\to\infty$ (see, example, [10])

$$\mathbf{E}(\mathbf{x}) - \mathbf{E}^i(\mathbf{x}) = \frac{e^{i\omega\|\mathbf{x}\|}}{\|\mathbf{x}\|}\mathbf{A}_\infty\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}; \mathbf{E}^i\right) + \mathcal{O}\left(\frac{1}{\|\mathbf{x}\|^2}\right), \tag{1.10}$$

where

$$\mathbf{A}_\infty(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d}) := \mathbf{A}_\infty\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}; \mathbf{E}^i\right) \tag{1.11}$$

is known as the *scattering amplitude* and $\hat{\mathbf{x}}$ denotes the direction of observation. It is readily verified that

$$\mathbf{A}_\infty(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d}) = \|\mathbf{p}\| \mathbf{A}_\infty\left(\hat{\mathbf{x}}; \frac{\mathbf{p}}{\|\mathbf{p}\|}, \mathbf{d}\right). \tag{1.12}$$

Therefore, without loss of generality and throughout the rest of our study, we shall assume that $\|\mathbf{p}\| = 1$, namely, $\mathbf{p} \in \mathbb{S}^2$.

Definition 1.1. Let $\Sigma_p \subset \mathbb{S}^2$, $\Sigma_d \subset \mathbb{S}^2$ and $\Sigma_{\hat{x}} \subset \mathbb{S}^2$. $\{\Omega; \varepsilon, \mu, \sigma\}$ is said to be a near/approximate-cloak if

$$\|\mathbf{A}_\infty(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})\| \ll 1 \quad \text{for} \quad \hat{\mathbf{x}} \in \Sigma_{\hat{x}}, \ \mathbf{p} \in \Sigma_p, \quad \mathbf{d} \in \Sigma_d. \tag{1.13}$$

If $\Sigma_p = \Sigma_d = \Sigma_{\hat{x}} = \mathbb{S}^2$, then it is called an approximate full-cloak, otherwise it is called an approximate partial-cloak.

According to Definition 1.1, the cloaking layer $\{\Omega\backslash\overline{\Omega}_c; \varepsilon_c^*, \mu_c^*, \sigma_c^*\}$ together with the conducting layer $\{\Omega_c\backslash\overline{\Omega}_a; \varepsilon_l^*, \mu_l^*, \sigma_l^*\}$ makes the target EM object $\{\Omega_a; \varepsilon_a^*, \mu_a^*, \sigma_a^*\}$ nearly invisible to detecting waves (1.6) with $\mathbf{d} \in \Sigma_d$ and $\mathbf{p} \in \Sigma_p$, and observation in the aperture $\Sigma_{\hat{x}}$. Σ_d and $\Sigma_{\hat{x}}$ shall be referred to as, respectively, the apertures of incidence and observation of the partial-cloaking device. For practical considerations, throughout the current study, we assume that the cloaking device is not object-dependent; that is, the cloaked content $\{\Omega_a; \varepsilon_a^*, \mu_a^*, \sigma_a^*\}$ is *arbitrary but regular*, namely, $\varepsilon_a^*, \mu_a^*, \sigma_a^*$ are all arbitrary symmetric positive definite matrices. In Definition 1.1, (1.13) is rather qualitative, and in the subsequent study, we shall quantify the near-cloaking effect and derive sharp estimate in assessing the cloaking performance. To that end, the following theorem plays a critical role (cf. [6,7]).

Theorem 1.1. *Let* $(\mathbf{E}, \mathbf{H}) \in H_{loc}(\text{curl}; \mathbb{R}^3)^2$ *be the (unique) pair of solutions to* (1.8)*. Define the* pull-back fields *by*

$$\mathbf{E}_\delta = (F_\delta)^*\mathbf{E} := (DF_\delta)^T \mathbf{E} \circ F_\delta, \quad \mathbf{H}_\delta = (F_\delta)^*\mathbf{H} := (DF_\delta)^T \mathbf{H} \circ F_\delta.$$

Then the pull-back fields $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in H_{loc}(\text{curl}; \mathbb{R}^3)^2$ *satisfy the following Maxwell equations*

$$\begin{cases} \nabla \times \mathbf{E}_\delta - i\omega\mu_0 \mathbf{H}_\delta = 0 & \text{in } \mathbb{R}^3\backslash\overline{D}_\delta, \\ \nabla \times \mathbf{H}_\delta + i\omega\varepsilon_0 \mathbf{E}_\delta = 0 & \text{in } \mathbb{R}^3\backslash\overline{D}_\delta, \\ \nabla \times \mathbf{E}_\delta - i\omega\mu_\delta \mathbf{H}_\delta = 0 & \text{in } D_\delta, \\ \nabla \times \mathbf{H}_\delta + i\omega\left(\varepsilon_\delta + i\frac{\sigma_\delta}{\omega}\right)\mathbf{E}_\delta = 0 & \text{in } D_\delta \end{cases} \tag{1.14}$$

subject to the Silver–Müller radiation condition:

$$\lim_{\|\mathbf{x}\|\to\infty} \|\mathbf{x}\|\left((\mathbf{H}_\delta - \mathbf{H}^i) \times \hat{\mathbf{x}} - (\mathbf{E}_\delta - \mathbf{E}^i)\right) = 0. \tag{1.15}$$

In Particular, since $F_\delta = \text{Identity}$ *in* $\mathbb{R}^3\backslash\Omega$*, one has that*

$$\mathbf{A}_\infty(\hat{\mathbf{x}}; \mathbf{E}^i) = \mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{E}^i), \quad \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{1.16}$$

where $\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{E}^i)$ *denotes the scattering amplitude corresponding to the Maxwell system* (1.14).

Hence, by Theorem 1.1, in order to assess the cloaking performance of the cloaking device $\{\Omega; \varepsilon, \mu, \sigma\}$ in (1.2) associated with the physical scattering system (1.8)–(1.9), it suffices for us to investigate the virtual scattering system (1.14)–(1.15). Here, it is noted for emphasis that by (1.5), one has

$$\{D_{\delta/2}; \varepsilon_a, \mu_a, \sigma_a\} = (F_\delta^{-1})_* \left\{\Omega_a; \varepsilon_a^*, \mu_a^*, \sigma_a^*\right\}. \tag{1.17}$$

Since $\{\Omega_a; \varepsilon_a^*, \mu_a^*, \sigma_a^*\}$ is arbitrary and regular and F_δ is bi-Lipschitz and orientation-preserving, it is clear that $\{D_{\delta/2}; \varepsilon_a, \mu_a, \sigma_a\}$ is also arbitrary and regular.

In summarizing our discussion so far, in order to construct a near-cloaking device, one needs to firstly select a suitable generating set Γ_0 and the corresponding virtual domain D_δ; and secondly the blowup transformation F_δ in (1.1); and thirdly the virtual conducting layer $\{D_\delta \backslash \overline{D}_{\delta/2}; \varepsilon_l, \mu_l, \sigma_l\}$; and finally assess the cloaking performance by studying the virtual scattering system (1.14)–(1.15). In this article, we shall be mainly concerned with extending the full- and partial-cloaking schemes proposed in [9,25] for acoustic waves to the more challenging case with electromagnetic waves. There the generating sets are either a generic curve or a planar subset, and the blowup transformations are constructed via a concatenating technique. Hence, in the current study, we shall mainly focus on properly designing the suitable conducting layers and then assessing the cloaking performances by studying the corresponding virtual scattering system (1.14)–(1.15).

The rest of the paper is organized as follows. In Section 2, we collect some preliminary knowledge on boundary layer potentials, which shall be used throughout our study. Sections 3 and 4 are, respectively, devoted to the study of the full- and partial-cloaking schemes.

2. Boundary Layer Potentials

Our study shall heavily rely on the vectorial boundary integral operators for Maxwell's equations. In this section, we review some of the important properties of the vectorial boundary integral operators for the later use.

2.1. Definitions

Let D be a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$-smooth boundary ∂D and a connected complement $\mathbb{R}^3 \backslash \overline{D}$. Let $\nabla_{\partial D} \cdot$ denote the surface divergence on ∂D and $\mathrm{H}^s(\partial D)$ be the usual Sobolev space of order $s \in \mathbb{R}$ on ∂D. Let ν be the exterior unit normal vector to ∂D and denote by $\mathrm{TH}^s(\partial D) := \{\mathbf{a} \in \mathrm{H}^s(\partial D)^3; \nu \cdot \mathbf{a} = 0\}$, the space of vectors tangential to ∂D which is a subset of $\mathrm{H}^s(\partial D)^3$. We also introduce the function space

$$\mathrm{TH}^s_{\mathrm{div}}(\partial D) := \left\{\mathbf{a} \in \mathrm{TH}^s(\partial D); \nabla_{\partial D} \cdot \mathbf{a} \in \mathrm{H}^s(\partial D)\right\},$$

endowed with the norm

$$\|\mathbf{a}\|_{\mathrm{TH}^s_{\mathrm{div}}(\partial D)} = \|\mathbf{a}\|_{\mathrm{TH}^s(\partial D)} + \|\nabla_{\partial D} \cdot \mathbf{a}\|_{\mathrm{H}^s(\partial D)}.$$

Next, we recall that, for $\omega \in \mathbb{R}_+ \cup \{0\}$, the fundamental outgoing solution G_ω to the PDO $(\Delta + \omega^2)$ in $\mathbb{R}^3$ is given by

$$G_\omega(\mathbf{x}) = -\frac{e^{i\omega\|\mathbf{x}\|}}{4\pi\|\mathbf{x}\|}. \tag{2.1}$$

In what follows, if $\omega = 0$ we simply write G_ω as G.

For a density function $\mathbf{a} \in \mathrm{TH}^s_{\mathrm{div}}(\partial D)$, we define the vectorial single layer potential associated with the fundamental solution G_ω introduced in (2.1) by

$$\mathcal{S}^\omega_D[\mathbf{a}](\mathbf{x}) := \int_{\partial D} G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\, \mathrm{d}\sigma_y, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.2}$$

For a scalar density $\varphi \in \mathrm{H}^s(\partial D)$, the single layer potential is defined similarly by

$$\mathcal{S}^\omega_D[\varphi](\mathbf{x}) := \int_{\partial D} G_\omega(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})\, \mathrm{d}\sigma_y, \quad \mathbf{x} \in \mathbb{R}^3. \tag{2.3}$$

The following boundary operator shall also be needed:

$$\begin{aligned} \mathcal{M}^\omega_D : \mathrm{L}^2_T(\partial D) &\longrightarrow \mathrm{L}^2_T(\partial D) \\ \mathbf{a} &\longrightarrow \mathcal{M}^\omega_D[\mathbf{a}](\mathbf{x}) = \text{p.v.} \quad \nu_\mathbf{x} \times \nabla \times \int_{\partial D} G_\omega(\mathbf{x},\mathbf{y})\mathbf{a}(\mathbf{y})\, \mathrm{d}\sigma_y, \end{aligned} \tag{2.4}$$

where $\mathrm{L}^2_T(\partial D) := \mathrm{TH}^0(\partial D)$, and p.v. signifies the Cauchy principle value. In what follows, we denote by $\mathcal{A}_D$, $\mathcal{S}_D$ and $\mathcal{M}_D$ the operators $\mathcal{A}^0_D$, $\mathcal{S}^0_D$ and $\mathcal{M}^0_D$, respectively.

2.2. Boundary Integral Identities

Here and throughout the rest of the paper, we make use of the following notation: for a function u defined on $\mathbb{R}^3 \backslash \partial D$, we denote

$$u|_\pm(\mathbf{x}) = \lim_{\tau \to +0} u(\mathbf{x} \pm \tau\nu(\mathbf{x})), \quad \mathbf{x} \in \partial D,$$

and

$$\left.\frac{\partial u}{\partial \nu}\right|_\pm(\mathbf{x}) = \lim_{\tau \to +0} \langle \nabla_\mathbf{x} u(\mathbf{x} \pm \tau\nu(\mathbf{x})), \nu(\mathbf{x})\rangle, \quad \mathbf{x} \in \partial D,$$

if the limits exist, where ν is the unit outward normal vector to ∂D.

It is known that the single layer potential $\mathcal{S}^\omega_D$ satisfies the trace formula (cf. [10,31])

$$\left.\frac{\partial}{\partial \nu}\mathcal{S}^\omega_D[\varphi]\right|_\pm = \left(\pm\frac{1}{2}I + (\mathcal{K}^\omega_D)^*\right)[\varphi] \quad \text{on} \quad \partial D, \tag{2.5}$$

where $(\mathcal{K}^\omega_D)^*$ is the L^2-adjoint of $\mathcal{K}^\omega_D$ and

$$\mathcal{K}^\omega_D[\varphi] := \text{p.v.} \int_{\partial D} \frac{\partial G_\omega(\mathbf{x}-\mathbf{y})}{\partial \nu(\mathbf{y})}\varphi(\mathbf{y})\, \mathrm{d}\sigma_y, \quad \mathbf{x} \in \partial D.$$

The jump relations in the following proposition are also known (see [10,31]).

Proposition 2.1. *Let* $\mathbf{a} \in \mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial D)$. *Then* $\mathcal{S}_D^{\omega}[\mathbf{a}]$ *is continuous on* $\mathbb{R}^3$ *and its* curl *satisfies the following jump formula,*

$$\nu \times \nabla \times \mathcal{S}_D^{\omega}[\mathbf{a}]\big|_{\pm} = \mp\frac{\mathbf{a}}{2} + \mathcal{M}_D^{\omega}[\mathbf{a}] \quad on \quad \partial D, \tag{2.6}$$

where

$$\nu(\mathbf{x}) \times \nabla \times \mathcal{S}_D^{\omega}[\mathbf{a}]\big|_{\pm}(\mathbf{x}) = \lim_{t\to+0} \nu(\mathbf{x}) \times \nabla \times \mathcal{S}_D^{\omega}[\mathbf{a}](\mathbf{x} \pm t\nu(\mathbf{x})), \quad \forall \mathbf{x} \in \partial D.$$

Equipped with the above knowledge, the solution pair $(\mathbf{E}_\delta, \mathbf{H}_\delta)$ in $\mathbb{R}^3\backslash\overline{D}_\delta$ to (1.14) can be represented using the following integral ansatz,

$$\mathbf{E}_\delta(\mathbf{x}) = \mathbf{E}^i(\mathbf{x}) + \nabla_{\mathbf{x}} \times \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{D}_\delta, \tag{2.7}$$

$$\mathbf{H}_\delta(\mathbf{x}) = \frac{1}{i\omega}\nabla_{\mathbf{x}} \times \mathbf{E}_\delta(\mathbf{x}) = \mathbf{H}^i(\mathbf{x}) + \frac{1}{i\omega}\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3\backslash\overline{D}_\delta,$$
z (2.8)

where by (2.6) the vectorial density function $\mathbf{a} \in \mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial D_\delta)$ satisfies

$$\Big(-\frac{I}{2} + \mathcal{M}_{D_\delta}^{\omega}\Big)[\mathbf{a}](\mathbf{x}) = \nu \times (\mathbf{E}_\delta - \mathbf{E}^i)(\mathbf{x})\Big|_{+}, \quad \mathbf{y} \in \partial D_\delta. \tag{2.9}$$

3. Regularized Full-Cloaking of EM Waves

In this section, we consider a regularized full-cloaking scheme of the EM waves by taking the generating set to be a generic curve. As discussed in the introduction, this scheme was considered in our earlier work [9] for acoustic waves. For self-containedness, we briefly discuss the generating set Γ_0 and the virtual domain D_δ for the proposed cloaking scheme in the sequel, which can also be found in [9]. Let Γ_0 be a smooth simple and non-closed curve in $\mathbb{R}^3$ with two endpoints, denoted by P_0 and Q_0, respectively. Denote by $N(\mathbf{x})$ the normal plane of the curve Γ_0 at $\mathbf{x} \in \Gamma_0$. We note that $N(P_0)$ and $N(Q_0)$ are, respectively, defined by the left and right limits along Γ_0. Let $q \in \mathbb{R}_+$. For any $\mathbf{x} \in \Gamma_0$, we let $\mathscr{S}_q(\mathbf{x})$ denote the disk lying on $N(\mathbf{x})$, centered at $\mathbf{x}$ and of radius q. It is assumed that there exists $q_0 \in \mathbb{R}_+$ such that when $q \leqq q_0$, $\mathscr{S}_q(\mathbf{x})$ intersects Γ_0 only at $\mathbf{x}$. We let D_q^f be given as

$$D_q^f := \mathscr{S}_q(\mathbf{x}) \times \Gamma_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\Gamma}_0, \tag{3.1}$$

where Γ_0 is identified with its parametric representation $\Gamma_0(\mathbf{x})$; see Fig. 1 for a schematic illustration. Clearly, the facade of D_q^f, denoted by S_q^f and parallel to Γ_0, is given by

$$S_q^f := \left\{\mathbf{x} + q \cdot \mathbf{n}(\mathbf{x}); \mathbf{x} \in \Gamma_0, \mathbf{n}(\mathbf{x}) \in N(\mathbf{x}) \cap \mathbb{S}^2\right\}, \tag{3.2}$$

and the two end-surfaces of D_q^f are the two disks $\mathscr{S}_q(P_0)$ and $\mathscr{S}_q(Q_0)$. Let D_q^a and D_q^b be two semi-balls of radius q and centred, respectively, at P_0 and Q_0. Let S_q^a and S_q^b be the two hemispheres such that $\partial D_q^a = S_q^a \cup \mathscr{S}_q(P_0)$ and $\partial D_q^b = S_q^b \cup \mathscr{S}_q(Q_0)$.

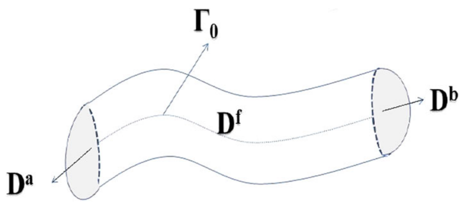

Fig. 1. Schematic illustration of the domain D_q for the regularized full-cloak

It is assumed that $S_q := S_q^f \cup S_q^b \cup S_q^a$ is a $C^{1,1}$-smooth boundary of the domain $D_q := D_q^a \cup D_q^f \cup D_q^b$.

Henceforth, we let $\delta \in \mathbb{R}_+$ be the asymptotically small regularization parameter and let D_δ denote the virtual domain used for the blowup construction of the cloaking device. We also let

$$S_\delta := S_\delta^f \cup S_\delta^b \cup S_\delta^a \tag{3.3}$$

denote the boundary surface of the virtual domain D_δ. Without loss of generality, we assume that $q_0 \equiv 1$. We shall drop the dependence on q if one takes $q = 1$. For example, D and S denote, respectively, D_q and S_q with $q = 1$. It is remarked that in all of our subsequent arguments, D can always be replaced by D_{τ_0}, with $0 < \tau_0 \leqq q_0$ being a fixed number.

In what follows, if we utilize $\mathbf{z}$ to denote the space variable on Γ_0, then for every $\mathbf{y} \in D_q^f$, we define a new variable $\mathbf{z}_y \in \Gamma_0$ which is the projection of $\mathbf{y}$ onto Γ_0. Meanwhile, if $\mathbf{y}$ belongs to D_q^a (respectively D_q^b), then $\mathbf{z}_y$ is defined to be P_0 (respectively Q_0). Henceforth, we let ξ denote the arc-length parameter of Γ_0 and θ, which ranges from 0 to 2π, be the angle of the point on $N(\xi)$ with respect to the central point $\mathbf{x}(\xi) \in \Gamma_0$. Moreover, we assume that if $\theta = 0$, then the corresponding points are those lying on the line that connects $\Gamma_0(\xi)$ to $\Gamma_1(\xi)$, where Γ_1 is defined to be

$$\Gamma_1 := \{\mathbf{x} + \mathbf{n}_1(\mathbf{x}); \mathbf{x} \in \Gamma_0, \mathbf{n}_1(\mathbf{x}) \in N(\mathbf{x}) \cap \mathbb{S}^2 - \text{a fixed vector for a given } \mathbf{x}\}.$$

With the above preparation, we introduce a blowup transformation which maps $\mathbf{y} \in \overline{D_\delta}$ to $\tilde{\mathbf{y}} \in \overline{D}$ as follows:

$$A(\mathbf{y}) = \tilde{\mathbf{y}} := \frac{1}{\delta}(\mathbf{y} - \mathbf{z}_y) + \mathbf{z}_y, \quad \mathbf{y} \in D_\delta^f, \tag{3.4}$$

whereas

$$A(\mathbf{y}) = \tilde{\mathbf{y}} := \begin{cases} \frac{\mathbf{y}-P_0}{\delta} + P_0, & \mathbf{y} \in D_\delta^a, \\ \frac{\mathbf{y}-Q_0}{\delta} + Q_0, & \mathbf{y} \in D_\delta^b. \end{cases} \tag{3.5}$$

Next, we present the crucial design of the lossy layer in (1.3). Define the Jacobian matrix $\mathbf{B}$ by

$$\mathbf{B}(\mathbf{y}) = \nabla_{\mathbf{y}} A(\mathbf{y}), \quad \mathbf{y} \in \overline{D_\delta}. \tag{3.6}$$

Set the material parameters ε_δ, μ_δ and σ_δ in the lossy layer $D_\delta \backslash \overline{D}_{\delta/2}$ to be

$$\begin{aligned} \varepsilon_\delta(\mathbf{x}) = \varepsilon_l(\mathbf{x}) :&= \delta^r |\mathbf{B}|\mathbf{B}^{-1}, \quad \mu_\delta(\mathbf{x}) = \mu_l(\mathbf{x}) := \delta^s |\mathbf{B}|\mathbf{B}^{-1}, \\ \sigma_\delta(\mathbf{x}) = \sigma_l(\mathbf{x}) :&= \delta^t |\mathbf{B}|\mathbf{B}^{-1}, \quad \text{for} \quad \mathbf{x} \in D_\delta \backslash \overline{D}_{\delta/2}, \end{aligned} \tag{3.7}$$

where r, s and t are all real numbers and $|\cdot|$ stands for the determinant when related to a square matrix.

We are now in a position to present the main theorem on the approximate full-cloak constructed by using D_δ described above as the virtual domain.

Theorem 3.1. *Let D_δ be as described above with $\partial D_\delta = S_\delta$ defined in (3.3). Let $(\mathbf{E}_\delta, \mathbf{H}_\delta)$ be the pair of solutions to (1.14), with $\{\Omega; \varepsilon_\delta, \mu_\delta, \sigma_\delta\} \subset \{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ defined in (1.3), and $\{D_\delta \backslash \overline{D}_{\delta/2}; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ given in (3.7). Define*

$$\beta = \min\{1, -1 + r + s, -1 + t + s\}, \quad \beta' = \min\{1, -2 + r + s, -2 + t + s\}.$$

If r, s and t are chosen such that $\beta' - t/2 \geqq 1/2$, then there exists $\delta_0 \in \mathbb{R}_+$ such that when $\delta < \delta_0$,

$$\|\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})\| \leqq C(\delta^{\beta - t/2 + 1} + \delta^2) \tag{3.8}$$

where C is a positive constant depending on ω and D, but independent of ε_a, μ_a, σ_a and $\hat{\mathbf{x}}$, $\mathbf{p}$, $\mathbf{d}$.

Remark 3.1. Following our earlier discussion, one can immediately infer by Theorems 1.1 and 3.1 that the push-forwarded structure in (1.2),

$$\{\Omega; \varepsilon, \mu, \sigma\} = (F_\delta)_* \{\Omega; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$$

produces an approximate full-cloaking device within at least δ-accuracy to the ideal cloak. Indeed, if we set $s = 3$, $r = 0$ and $t = -1$ then one has $\beta = 1$, $\beta' = 0$ and the accuracy of the ideal cloak will be δ^2, which is the highest accuracy that one can obtain for such a construction. In Particular, it is emphasized that in (3.8), the estimate is independent of ε_a, μ_a, σ_a, and this means that the cloaked content $\{\Omega_a; \varepsilon_a^*, \mu_a^*, \sigma_a^*\}$ in (1.2) can be arbitrary but regular. Finally, as remarked earlier, we refer to [25] for the construction of the blowup transformation F_δ, of which we always assume the existence in the current study.

The subsequent three subsections are devoted to the proof of Theorem 3.1. For our later use, we first derive some critical lemmas.

3.1. Auxiliary Lemmas

In this subsection we present some auxiliary lemmas that are essential for our analysis of the far-field estimates. To begin with, we show the following properties of the blowup transformation defined in (3.4).

Lemma 3.1. (Lemma 4.1 in [9]) *Let A be the transformation introduced in* (3.4) *and* (3.5) *which maps the region $\overline{D_\delta}$ to $\overline{D}$. Let $\mathbf{B}(\mathbf{y})$ be the corresponding Jacobian matrix of $A(\mathbf{y})$ given by* (3.6)*. Then we have*

$$\mathbf{B}(\mathbf{y}) = \begin{cases} \frac{1}{\delta}\mathbf{I}_{3\times 3} - \left(\frac{1}{\delta} - 1\right)\mathbf{z}'_y(\xi)\mathbf{z}'_y(\xi)^T, & \mathbf{y} \in D_\delta^f, \\ \frac{1}{\delta}\mathbf{I}_{3\times 3}, & \mathbf{y} \in D_\delta^a \cup D_\delta^b, \end{cases} \tag{3.9}$$

where the superscript T denotes the transpose of a vector or a matrix. Furthermore,

$$\mathbf{B}(\mathbf{y})\nu_{\mathbf{y}} = \frac{1}{\delta}\nu_{\mathbf{y}}, \quad \mathbf{y} \in \partial D_\delta, \tag{3.10}$$

where $\nu_{\mathbf{y}}$ stands for the unit outward normal vector to ∂D_δ at $\mathbf{y} \in \partial D_\delta$.

Remark 3.2. In view of the Jacobian matrix form (3.9), one can also find that the eigenvalues of $\mathbf{B}(\mathbf{x})$, $\mathbf{x} \in D_\delta$ are either 1 or $1/\delta$. Hence for any vector field $\mathbf{V} \in \mathbb{R}^3$, there holds

$$\|\mathbf{V}\|^2 \leqq \langle \mathbf{B}(\mathbf{x})\mathbf{V}, \mathbf{V}\rangle \leqq \delta^{-1}\|\mathbf{V}\|^2 \tag{3.11}$$

uniformly for $\mathbf{x} \in D_\delta$. It can also be easily seen from (3.9) that

$$|\mathbf{B}(\mathbf{y})| = \delta^{-2}, \quad \mathbf{y} \in D_\delta^f. \tag{3.12}$$

For the sake of simplicity, we define

$$\mathbf{E}_\delta^+ := \mathbf{E}_\delta - \mathbf{E}^i, \quad \mathbf{H}_\delta^+ := \mathbf{H}_\delta - \mathbf{H}^i, \quad \text{in} \quad \mathbb{R}^3\backslash\overline{D}_\delta. \tag{3.13}$$

Furthermore, we introduce the following notations

$$\tilde{\mathbf{E}}(\tilde{\mathbf{x}}) := \mathbf{E}(A^{-1}(\tilde{\mathbf{x}})) = \mathbf{E}(\mathbf{x}), \quad \tilde{\mathbf{H}}(\tilde{\mathbf{x}}) := \mathbf{H}(A^{-1}(\tilde{\mathbf{x}})) = \mathbf{H}(\mathbf{x}), \tag{3.14}$$

and define the corresponding fields after change of variables by

$$\hat{\mathbf{E}}(\tilde{\mathbf{x}}) := ((\tilde{\mathbf{B}}^T)^{-1}\tilde{\mathbf{E}})(\tilde{\mathbf{x}}), \quad \hat{\mathbf{H}}(\tilde{\mathbf{x}}) := ((\tilde{\mathbf{B}}^T)^{-1}\tilde{\mathbf{H}})(\tilde{\mathbf{x}}), \tag{3.15}$$

where

$$\tilde{\mathbf{B}}(\tilde{\mathbf{x}}) := \mathbf{B}(A^{-1}(\tilde{\mathbf{x}})) = \mathbf{B}(\mathbf{x}).$$

We mention that sometimes we write $\mathbf{B}$ and $\tilde{\mathbf{B}}$ in the sequel and omit their dependences for simplicity. The following lemma is of critical importance for our subsequent analysis.

Lemma 3.2. (Corollary 3.58 in [32]) *Let $\tilde{\mathbf{x}} = A(\mathbf{x})$ with $\mathbf{B}(\mathbf{x}) = \nabla A(\mathbf{x})$. Then for the bounded domain D_δ and any vector field $\mathbf{V} \in H(\text{curl}; D_\delta)$,*

$$\hat{\mathbf{V}}(\tilde{\mathbf{x}}) := (\tilde{\mathbf{B}}^T)^{-1}\tilde{\mathbf{V}}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{V}}(\tilde{\mathbf{x}}) := \mathbf{V}(\mathbf{x}), \quad \mathbf{x} \in D_\delta,$$

there hold the following identities

$$|\tilde{\mathbf{B}}|^{-1}\tilde{\mathbf{B}}(\nabla \times \mathbf{V})(A^{-1}(\tilde{\mathbf{x}})) = \nabla_{\tilde{\mathbf{x}}} \times \hat{\mathbf{V}}(\tilde{\mathbf{x}}), \tag{3.16}$$

and

$$\int_{\partial D_\delta} (\nu_{\mathbf{x}} \times \mathbf{V}) \cdot \mathbf{W} d\sigma_x = \int_{\partial D} (\nu_{\tilde{\mathbf{x}}} \times \hat{\mathbf{V}}) \cdot \hat{\mathbf{W}} d\sigma_{\tilde{x}} \tag{3.17}$$

where $\mathbf{W} \in H(\text{curl}; D_\delta)$ and $\hat{\mathbf{W}}(\tilde{\mathbf{x}}) := (\tilde{\mathbf{B}}^T)^{-1}\tilde{\mathbf{W}}(\tilde{\mathbf{x}}) := (\mathbf{B}^T)^{-1}\mathbf{W}(\mathbf{x})$.

Noting that

$$d\sigma_y = \begin{cases} \delta\, d\sigma_{\tilde{y}}, & \mathbf{y} \in S_\delta^f, \\ \delta^2\, d\sigma_{\tilde{y}}, & \mathbf{y} \in S_\delta^b \cup S_\delta^a, \end{cases} \tag{3.18}$$

with which one can show that the following Lemma applies

Lemma 3.3. *Let D^c, $c \in \{a, b\}$ be defined at the beginning of this section. For $r \in \mathbb{R}_+$, let $D^a_{r,\delta}$ denote the semi-spheroid with $\mathscr{S}_r(P_0)$ being its base and the other semi-axis being δ. Let $D^b_{r,\delta}$ be defined similarly with the base being $\mathscr{S}_r(Q_0)$. Define $D_{1,\delta} := D^f \cup D^a_{1,\delta} \cup D^b_{1,\delta}$ and denote by A_1 the blowup transformation from $\mathbf{x} \in D_\delta$ to $\tilde{\mathbf{x}} \in D_{1,\delta}$ with $A_1\big|_{D_\delta^f} = A\big|_{D_\delta^f}$. Let $\mathbf{V}$ and $\hat{\mathbf{V}}$, $\mathbf{B}$ be similarly defined as those in Lemma 3.2. Then for any $\mathbf{W} \in H(\mathrm{curl}; D)$, one has*

$$\int_{\partial D_{1,\delta}} \nu_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{V}} \cdot \mathbf{W}(\tilde{\mathbf{x}})\, \mathrm{d}\sigma_{\tilde{x}} = \delta^{-1} \int_{\partial D_{1,\delta}} \nu_{\tilde{\mathbf{x}}} \times \hat{\mathbf{V}} \cdot \hat{\mathbf{W}}(\tilde{\mathbf{x}})\, \mathrm{d}\sigma_{\tilde{x}}, \tag{3.19}$$

$$\int_{\partial D^c} \nu_{\tilde{\mathbf{x}}} \times \tilde{\mathbf{V}} \cdot \mathbf{W}(\tilde{\mathbf{x}})\, \mathrm{d}\sigma_{\tilde{x}} = \delta^{-2} \int_{\partial D^c} \nu_{\tilde{\mathbf{x}}} \times \hat{\mathbf{V}} \cdot \hat{\mathbf{W}}(\tilde{\mathbf{x}})\, \mathrm{d}\sigma_{\tilde{x}} \quad c \in \{a, b\}, \tag{3.20}$$

where $\hat{\mathbf{W}}(\tilde{\mathbf{x}}) := (\tilde{\mathbf{B}}^T)^{-1}\mathbf{W}(\tilde{\mathbf{x}})$.

Proof. The proof follows directly from (3.17), (3.18) by using a change of variables in the corresponding integrals. □

Lemma 3.4. *Suppose $\tilde{\mathbf{a}}(\tilde{\mathbf{x}}) = \mathbf{a}(\mathbf{x})$ for $\mathbf{x} \in \partial D_\delta$ and $\tilde{\mathbf{x}} \in \partial D$. Let $\iota_{\delta^t}(\mathbf{x})$ and $\iota_{1,\delta^t}(\tilde{\mathbf{x}})$, $t = 1/2$ or 1, be two regions respectively defined by*

$$\iota_{\delta^t}(\mathbf{x}) := \left\{ \mathbf{y} \Big| |\mathbf{z}_x - \mathbf{z}_y| < \delta^t, \mathbf{y} \in \partial D_\delta \right\},$$
$$\iota_{1,\delta^t}(\tilde{\mathbf{x}}) := \left\{ \tilde{\mathbf{y}} \Big| |\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}| < \delta^t, \tilde{\mathbf{y}} \in \partial D \right\}.$$

Define

$$\mathcal{M}^\omega_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) := \mathrm{p.v.} \quad \nu_{\tilde{\mathbf{x}}} \times \nabla_{\mathbf{z}_{\tilde{x}}} \times \int_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}} G_\omega(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}) \tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, \mathrm{d}\sigma_{\tilde{y}},$$

and

$$\mathcal{M}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) := \mathrm{p.v.} \quad -\frac{1}{4\pi} \nu_{\tilde{\mathbf{x}}} \times \int_{S^c \cap \overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} \frac{\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})}{\|\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\|^3} \times \tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, \mathrm{d}\sigma_{\tilde{y}},$$

where

$$\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \tilde{\mathbf{x}} - \tilde{\mathbf{y}} + (1/\delta - 1)(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}).$$

Then there holds the following result:

$$\mathcal{M}^\omega_{D_\delta}[\mathbf{a}](\mathbf{x}) = \delta \mathcal{M}^\omega_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathcal{M}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathscr{R}_1(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}), \tag{3.21}$$

where the remainder term $\mathscr{R}_1(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})$ *satisfies*

$$\lim_{\delta\to 0}\|\mathscr{R}_1(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})\|/(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})=0. \tag{3.22}$$

Define

$$\mathcal{L}^{\omega}_{D_\delta}(\mathbf{x}):=\nu_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\mathcal{S}^{\omega}_{D_\delta}[\mathbf{a}](\mathbf{x}),$$

then there holds

$$\mathcal{L}^{\omega}_{D_\delta}[\mathbf{a}](\mathbf{x})=\delta\mathcal{L}^{\omega}_{S^f\backslash\overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})+\frac{1}{\delta}\mathcal{A}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})+\mathscr{R}_2(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}), \tag{3.23}$$

where

$$\mathcal{L}^{\omega}_{S^f\backslash\overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\mathbf{x}):=\nu_{\tilde{\mathbf{x}}}\times\nabla_{\mathbf{z}_{\tilde{x}}}\times\nabla_{\mathbf{z}_{\tilde{x}}}\times\int_{S^f\backslash\overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}G_\omega(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}},$$

and

$$\mathcal{A}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})=-\frac{1}{4\pi}\nu_{\tilde{\mathbf{x}}}\times\int_{S^c\cap\iota_{1,\delta}(\tilde{\mathbf{x}})}\frac{1}{\|\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\|^3}\left(\frac{\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})^T}{\|\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\|^2}-I\right)\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}},$$

and the remainder term $\mathscr{R}_2(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})$ *satisfies* (3.22).

Proof. We shall follow the similar strategy of Lemma 4.2 in [9]. Let $\mathbf{x}\in\partial D_\delta$ and we separate ∂D^δ into four different regions:

$$\partial D_\delta=\left(\partial D_\delta\backslash\overline{\iota_{\delta^{1/2}}(\mathbf{x})}\right)\cup\left((S^a_\delta\cup S^b_\delta)\cap\overline{\iota_\delta(\mathbf{x})}\right)\cup\left(S^f_\delta\cap\overline{\iota_\delta(\mathbf{x})}\right)\cup\left(\iota_{\delta^{1/2}}(\mathbf{x})\backslash\iota_\delta(\mathbf{x})\right).$$

We consider $\mathbf{y}\in\partial D^\delta$ in those four different regions, respectively. We compute

$$\begin{aligned}\mathcal{M}^{\omega}_{D_\delta}[\mathbf{a}](\mathbf{x})&=\nu_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\int_{\partial D^f_\delta\backslash\overline{\iota_{\delta^{1/2}}(\tilde{\mathbf{x}})}}G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\&\quad+\nu_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\int_{S^c_\delta\cap\overline{\iota_\delta(\tilde{\mathbf{x}})}}G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\&\quad+\nu_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\int_{S^f_\delta\cap\overline{\iota_\delta(\tilde{\mathbf{x}})}}G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\&\quad+\nu_{\mathbf{x}}\times\nabla_{\mathbf{x}}\times\int_{\iota_{\delta^{1/2}}(\tilde{\mathbf{x}})\cap\iota_\delta(\tilde{\mathbf{x}})}G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\&:=I_1+I_2+I_3+I_4.\end{aligned}$$

In the following, we shall estimate I_1, I_2, I_3 and I_4 separately. For $\mathbf{x},\mathbf{y}\in\partial D_\delta$, one has

$$\mathbf{x}-\mathbf{y}=(\mathbf{x}-\mathbf{z}_x)-(\mathbf{y}-\mathbf{z}_y)+\mathbf{z}_x-\mathbf{z}_y=\delta((\tilde{\mathbf{x}}-\mathbf{z}_{\tilde{x}})-(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}}))+(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}).$$

Hence, we have the following expansion for $\mathbf{y}\in\partial D^f_\delta\backslash\overline{\iota_{\delta^{1/2}}(\mathbf{x})}$,

$$\|\mathbf{x}-\mathbf{y}\|=\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|+\delta\left\langle(\tilde{\mathbf{x}}-\mathbf{z}_{\tilde{x}})-(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}}),\frac{\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}}{\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|}\right\rangle+\mathcal{O}(\delta^{3/2}).$$

Similarly,

$$\begin{aligned}
&\langle \mathbf{x}-\mathbf{y}, \nu_{\mathbf{x}}\rangle = \langle \mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}, \nu_{\mathbf{x}}\rangle + \delta\langle (\tilde{\mathbf{x}}-\mathbf{z}_{\tilde{x}})-(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}}), \nu_{\mathbf{x}}\rangle,\\
&\|\mathbf{x}-\mathbf{y}\|^{-1} = \|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|^{-1} - \delta\Big\langle (\tilde{\mathbf{x}}-\mathbf{z}_{\tilde{x}})-(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}}), \frac{\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}}{\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|^3}\Big\rangle + \mathcal{O}(\delta^{3/2}),\\
&e^{i\omega\|\mathbf{x}-\mathbf{y}\|} = e^{i\omega\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|}\Big(1+i\omega\delta\Big\langle (\tilde{\mathbf{x}}-\mathbf{z}_{\tilde{x}})-(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}}), \frac{\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}}{\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|}\Big\rangle\Big) + \mathcal{O}(\delta^{3/2}).
\end{aligned}$$

With those expansions to hand, we proceed to compute for $\mathbf{y} \in \partial D_\delta^f \setminus \overline{\iota_{\delta^{1/2}}(\mathbf{x})}$,

$$\begin{aligned}
\nabla_{\mathbf{x}} G_\omega(\mathbf{x}-\mathbf{y}) &= \frac{(\mathbf{x}-\mathbf{y})e^{i\omega\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|^2}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - i\omega\right)\\
&= \frac{(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})e^{i\omega\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|}}{4\pi\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|^2}\left(\frac{1}{\|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}\|} - i\omega\right) + \mathcal{O}(\delta)\\
&= \nabla_{\mathbf{z}_{\tilde{x}}} G_\omega(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}) + \mathcal{O}(\delta).
\end{aligned}$$

By using vector calculus identities, one has

$$\begin{aligned}
I_1 &= \int_{\partial D_\delta^f \setminus \overline{\iota_{\delta^{1/2}}(\tilde{\mathbf{x}})}} \nabla_{\mathbf{x}} G_\omega(\mathbf{x}-\mathbf{y})\nu_{\mathbf{x}} \cdot \mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\
&\quad - \int_{\partial D_\delta^f \setminus \overline{\iota_{\delta^{1/2}}(\tilde{\mathbf{x}})}} \nu_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\
&= \delta\int_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}} \nabla_{\mathbf{z}_{\tilde{x}}} G_\omega(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\nu_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\\
&\quad - \delta\int_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}} \nu_{\tilde{\mathbf{x}}} \cdot \nabla_{\mathbf{z}_{\tilde{x}}} G_\omega(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}} + \mathcal{O}(\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})\\
&= \delta\mathcal{M}^\omega_{S^f \setminus \overline{\iota_{1,\delta^{1/2}}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathcal{O}(\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}).
\end{aligned}$$

Next, note that if $\mathbf{y} \in S_\delta^c \cap \overline{\iota_\delta(\mathbf{x})}$, $c \in \{a, b\}$, then

$$\frac{\langle \mathbf{x}-\mathbf{y}, \nu_{\mathbf{x}}\rangle}{\|\mathbf{x}-\mathbf{y}\|} = \frac{\langle \tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}), \nu_{\mathbf{x}}\rangle}{\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\|},$$

and therefore

$$\begin{aligned}
I_2 &= \int_{S_\delta^c \cap \overline{\iota_\delta(\tilde{\mathbf{x}})}} \nabla_{\mathbf{x}} G_\omega(\mathbf{x}-\mathbf{y})\nu_{\mathbf{x}} \cdot \mathbf{a}(\mathbf{y})\,d\sigma_y - \int_{S_\delta^c \cap \overline{\iota_\delta(\tilde{\mathbf{x}})}} \nu_{\mathbf{x}} \cdot \nabla_{\mathbf{x}} G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\\
&= -\frac{1}{4\pi}\int_{S^c \cap \overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} \frac{\tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})}{\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\|^3}\nu_{\tilde{\mathbf{x}}} \cdot \tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\\
&\quad + \frac{1}{4\pi}\int_{S^c \cap \overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} \frac{\langle \tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}), \nu_{\mathbf{x}}\rangle}{\|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}+(1/\delta-1)(\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}})\|}\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\\
&= \mathcal{M}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})
\end{aligned}$$

By using exactly the same strategy in Lemma 4.2 in [9], one can show that I_3 and I_4 can be absorbed into the remainder term $\mathscr{R}_1(\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})$.

To prove (3.23), we first note that

$$\nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) = \omega^2 \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) + \nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \cdot \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}). \tag{3.24}$$

By using a similar analysis to these above for the two terms in the RHS of (3.24), one can show (3.23).

The proof is complete. □

3.2. Asymptotic Expansions

In order to tackle the integral equation (2.9), we first derive some crucial asymptotic expansions. Henceforth, we denote $\tilde{\mathbf{a}}(\tilde{\mathbf{y}}) := \mathbf{a}(\mathbf{y})$ for $\tilde{\mathbf{y}} = A(\mathbf{y})$, $\mathbf{y} \in D_\delta$ and $\tilde{\mathbf{y}} \in \partial D$. The same notation shall be adopted for $\mathbf{x}$ and $\tilde{\mathbf{x}}$.

For $\mathbf{x} \in \mathbb{R}^3 \backslash D_\delta$ with sufficiently large $\|\mathbf{x}\|$, we can expand $\mathbf{E}_\delta - \mathbf{E}^i$ from (2.7) in $\mathbf{z} \in \Gamma_0$ as follows:

$$\begin{aligned}
(\mathbf{E}_\delta - \mathbf{E}^i)(\mathbf{x}) &= \nabla \times \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) = \nabla \times \int_{\partial D_\delta} G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\, d\sigma_y \\
&= \nabla \times \int_{\partial D_\delta} G_\omega(\mathbf{x}-\mathbf{z}_y)\mathbf{a}(\mathbf{y})\, d\sigma_y - \nabla \times \int_{\partial D_\delta} \nabla G_\omega(\mathbf{x}-\mathbf{z}_y)\cdot(\mathbf{y}-\mathbf{z}_y)\mathbf{a}(\mathbf{y})\, d\sigma_y \\
&\quad + \nabla \times \int_{\partial D_\delta} (\mathbf{y}-\mathbf{z}_y)^T \nabla^2 G_\omega(\mathbf{x}-\zeta(\mathbf{y}))(\mathbf{y}-\mathbf{z}_y)\mathbf{a}(\mathbf{y})\, d\sigma_y \\
&:= R_1 + R_2 + R_3,
\end{aligned} \tag{3.25}$$

where $\zeta(\mathbf{y}) = \eta\mathbf{y} + (1-\eta)\mathbf{z}_y \in D_\delta$ for some $\eta \in (0,1)$, and the superscript T signifies the matrix transpose. We next estimate the three terms R_1, R_2 and R_3 in (3.25). The term R_3 in (3.25) is a remainder term from the Taylor series expansion and it verifies the following estimate:

$$\begin{aligned}
\|R_3\|_{L^\infty(\mathbb{S}^2)^3} &= \delta^2 \left\| \nabla \times \int_{\partial D_\delta} (\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})^T \nabla^2 G_\omega(\mathbf{x}-\zeta(\mathbf{y}))(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\mathbf{a}(\mathbf{y})\, d\sigma_y \right\|_{L^\infty(\mathbb{S}^2)^3} \\
&\leqq C\delta^3 \frac{1}{\|\mathbf{x}\|} \|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}.
\end{aligned} \tag{3.26}$$

In the sequel, we shall need the expansion of the incident plane wave $\mathbf{E}^i$ in $\mathbf{z} \in \Gamma_0$, and there holds

$$\mathbf{E}^i(\mathbf{y}) = \mathbf{E}^i(\mathbf{z}_y) + \nabla\mathbf{E}^i(\mathbf{z}_y)\cdot(\mathbf{y}-\mathbf{z}_y) + \sum_{|\alpha|=2}^{\infty} \partial_{\mathbf{y}}^{\alpha}\mathbf{E}^i(\mathbf{z}_y)(\mathbf{y}-\mathbf{z}_y)^{\alpha}, \tag{3.27}$$

where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\partial_{\mathbf{y}}^{\alpha} = \partial_{y_1}^{\alpha_1}\partial_{y_2}^{\alpha_2}\partial_{y_3}^{\alpha_3}$ with $\mathbf{y} = (y_1, y_2, y_3)$. Since for $\mathbf{y} \in \partial D_\delta$, $\nu_{\mathbf{y}} = \nu_{\tilde{\mathbf{y}}}$, one further has that

$$\nu_{\mathbf{y}} \times \mathbf{E}^i(\mathbf{y}) = \nu_{\tilde{\mathbf{y}}} \times \sum_{|\alpha|=0}^{\infty} \delta^{\alpha}\partial_{\mathbf{y}}^{\alpha}\mathbf{E}^i(\mathbf{z}_y)(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})^{\alpha}.$$

In what follows, we define

$$\tilde{\Phi}(\tilde{\mathbf{y}}) := \Phi(\mathbf{y}) = \nu_{\mathbf{y}} \times \mathbf{E}_\delta(\mathbf{y})\Big|_{\partial D_\delta}^{+}. \tag{3.28}$$

Theorem 3.2. *Let $\mathbf{E}_\delta$ be the solution to (1.14), then there holds for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$,*

$$\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}} = -2\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\Phi}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}} + \mathcal{O}(\delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}+1)). \tag{3.29}$$

If one assumes that $\Phi(\mathbf{y}) = 0$, $\mathbf{y} \in \partial D_\delta$, then there holds for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$,

$$\begin{aligned}(\mathbf{E}_\delta - \mathbf{E}^i)(\mathbf{x}) = 2\delta^2 \nabla \times \Bigg(&\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\nu_{\tilde{\mathbf{y}}} \times \big(\nabla \mathbf{E}^i(\mathbf{z}_{\tilde{y}})(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big)\,\mathrm{d}\sigma_y \\ &+ \int_{S^a \cup S^b} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\nu_{\tilde{\mathbf{y}}} \times \mathbf{E}^i(\mathbf{z}_{\tilde{y}})\,\mathrm{d}\sigma_{\tilde{y}} \\ &- \int_{S^f} \big(\nabla G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}}) \cdot (\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big)\nu_{\tilde{\mathbf{y}}} \times \mathbf{E}^i(\mathbf{z}_{\tilde{y}})(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\Bigg) + \mathcal{O}(\delta^3).\end{aligned}$$

Proof. Recall that $-\frac{I}{2} + \mathcal{M}_{D_\delta}^{\omega}$ is invertible on $\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial D_\delta)$ (see, example [18]). By (2.9), we see that

$$\mathbf{a}(\mathbf{x}) = \left(-\frac{I}{2} + \mathcal{M}_{D_\delta}^{\omega}\right)^{-1}\left[\nu_{\mathbf{y}} \times (\mathbf{E}_\delta - \mathbf{E}^i)(\mathbf{y})\Big|_{\partial D_\delta}^{+}\right](\mathbf{x}).$$

Using the results in Lemma 3.4, the invertibility of $-\frac{I}{2} + \mathcal{M}_{S^c}$ on $\mathrm{TH}_{\mathrm{div}}^{-1/2}(S^c)$ (the invertibility can be shown by following a complete similar argument to the proof of Theorem 4.2 in [9], along with the use of Proposition 4.7 in [2]), and the expansion of $\mathbf{E}^i$ in (3.27), we have for $\mathbf{x} \in S_\delta^f \backslash \overline{\iota_{1,\delta}(P_0) \cup \iota_{1,\delta}(Q_0)}$ that

$$\tilde{\mathbf{a}}(\tilde{\mathbf{x}}) = -2\tilde{\Phi}(\tilde{\mathbf{x}}) + 2\nu_{\tilde{\mathbf{x}}} \times \mathbf{E}^i(\mathbf{z}_{\tilde{x}}) + \mathcal{O}(\delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)), \tag{3.30}$$

and for $\mathbf{x} \in S_\delta^f \cap (\iota_{1,\delta}(P_1) \cup \iota_{1,\delta}(Q_0))$ we have

$$\tilde{\mathbf{a}}(\tilde{\mathbf{x}}) = -2\tilde{\Phi}(\tilde{\mathbf{x}}) + \mathcal{O}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1). \tag{3.31}$$

Noting that

$$\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})(\nu_{\tilde{\mathbf{y}}} \times \mathbf{E}^i(\mathbf{z}_{\tilde{y}}))\,\mathrm{d}\sigma_{\tilde{y}} = \int_{\Gamma_0} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\int_0^{2\pi} \nu_{\tilde{\mathbf{y}}}\,\mathrm{d}\vartheta \times \mathbf{E}^i(\mathbf{z}_{\tilde{y}})\,\mathrm{d}r = 0, \tag{3.32}$$

where (r, ϑ) stand for the polar coordinates, and this together with (3.30) and (3.31) readily implies (3.29).

Next, if $\tilde{\Phi}(\tilde{\mathbf{y}}) = \Phi(\mathbf{y}) = 0$, then it can be seen from (3.21), (3.30) and (3.31) that

$$\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-3/2}(\partial D)} \leqq C,$$

where C is a positive constant depending only on D and ω. By using our earlier results in (3.25) and (3.26), one can first show that

$$\begin{aligned}\mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) &= \int_{\partial D_\delta} G_\omega(\mathbf{x}-\mathbf{y})\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y \int_{\partial D_\delta} G_\omega(\mathbf{x}-\mathbf{z}_y)\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y \\ &= -\int_{\partial D_\delta} \big(\nabla G_\omega(\mathbf{x}-\mathbf{z}_y)\cdot(\mathbf{y}-\mathbf{z}_y)\big)\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y + \mathcal{O}(\delta^3) \\ &= \delta\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\mathrm{d}\sigma_{\tilde{y}} + \delta^2\Bigg(\int_{S^a\cup S^b} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}} \\ &\quad -\int_{S^f}\big(\nabla G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\cdot(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big)\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\Bigg) + \mathcal{O}(\delta^3). \end{aligned} \tag{3.33}$$

By using (3.30) we have for $\mathbf{y}\in S_\delta^f$ that

$$\tilde{\mathbf{a}}(\tilde{\mathbf{y}}) = 2\nu_{\tilde{\mathbf{y}}}\times\mathbf{E}^i(\mathbf{z}_{\tilde{y}}) + 2\delta\nu_{\tilde{\mathbf{y}}}\times\big(\nabla\mathbf{E}^i(\mathbf{z}_{\tilde{y}})(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big) + \mathcal{O}(\delta^2). \tag{3.34}$$

Substituting (3.34) into (3.33) and using (3.32) one can easily obtain

$$\begin{aligned}\mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) &= 2\delta^2\Bigg(\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_y)\nu_{\tilde{\mathbf{y}}}\times\big(\nabla\mathbf{E}^i(\mathbf{z}_{\tilde{y}})(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big)\,\mathrm{d}\sigma_y \\ &\quad + \int_{S^a\cup S^b} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\nu_{\tilde{\mathbf{y}}}\times\mathbf{E}^i(\mathbf{z}_{\tilde{y}})\,\mathrm{d}\sigma_{\tilde{y}} \\ &\quad - \int_{S^f}\big(\nabla G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\cdot(\tilde{\mathbf{y}}-\mathbf{z}_{\tilde{y}})\big)\nu_{\tilde{\mathbf{y}}}\times\mathbf{E}^i(\mathbf{z}_{\tilde{y}})(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\Bigg) + \mathcal{O}(\delta^3),\end{aligned}$$

which then completes the proof by using (2.7). □

We continue with the estimates of R_1 and R_2 in (3.25) for the proposed full-cloaking structure with arbitrary but regular ε_a, μ_a and σ_a in $D_{\delta/2}$. In what follows, we let C denote a generic positive constant. It may change from one inequality to another inequality in our estimates. Moreover, it may depend on different parameters, but it does not depend on ε_a, μ_a, σ_a and $\mathbf{p}$, $\mathbf{d}$, $\hat{\mathbf{x}}$.

We first note that, by using (3.30) and (3.31), there holds

$$\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)} \leqq C(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1). \tag{3.35}$$

Next, by taking expansion around $\mathbf{z}\in\Gamma_0$ and using (3.29) one can show that for $\delta\in\mathbb{R}_+$ sufficiently small and $\|\mathbf{x}\|$ sufficiently large,

$$\begin{aligned}\|R_1\| &= \left\|\nabla\times\int_{\partial D_\delta} G_\omega(\mathbf{x}-\mathbf{z}_y)\mathbf{a}(\mathbf{y})\,\mathrm{d}\sigma_y\right\| \\ &\leqq \delta\left\|\nabla\times\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\right\| + C\delta^2\frac{1}{\|\mathbf{x}\|}\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)} \\ &\leqq C\delta\left\|\nabla\times\int_{S^f} G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\tilde{\Phi}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}}\right\| + C\delta^2\frac{1}{\|\mathbf{x}\|}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1) \\ &\leqq C\delta\frac{1}{\|\mathbf{x}\|}\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + C\delta^2\frac{1}{\|\mathbf{x}\|}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1).\end{aligned} \tag{3.36}$$

By using Taylor's expansions again and (3.30), one can show the following estimation for R_2:

$$\|R_2\| = \left\| \nabla \times \int_{\partial D_\delta} \nabla G_\omega(\mathbf{x} - \mathbf{z}_y) \cdot (\mathbf{y} - \mathbf{z}_y)\mathbf{a}(\mathbf{y})\, \mathrm{d}\sigma_y \right\|$$
$$\leqq C\delta^2 \frac{1}{\|\mathbf{x}\|}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + 1) + C\delta^3 \frac{1}{\|\mathbf{x}\|}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1). \tag{3.37}$$

Hence, by applying the estimates in (3.26), (3.36) and (3.37) to (3.25), we have

$$\|\mathbf{E}_\delta - \mathbf{E}^i\| \leqq C \frac{\delta}{\|\mathbf{x}\|} \|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + C\delta^2 \frac{1}{\|\mathbf{x}\|}(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1) \tag{3.38}$$

for $\|\mathbf{x}\|$ sufficiently large. With (3.38) at hand, one readily has

Lemma 3.5. *The scattering amplitude $\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})$ corresponding to the scattering configuration described in Theorem 3.1 satisfies*

$$\|\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})\| \leqq C\delta \|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + C\delta^2(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1), \tag{3.39}$$

where C depends only on D and ω.

3.3. Proof of Theorem 3.1

By Lemma 3.5, it is straightforward to see that in order to derive the estimate of the scattering amplitude $\mathbf{A}_\infty^\delta$, it suffices for us to derive the corresponding estimates of $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)}$ and $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}$. To that end, we shall need the following lemma, whose proof can be found in [7].

Lemma 3.6. *Let B_R be a central ball of radius R such that $D_\delta \Subset B_R$. Then the solutions $E_\delta, H_\delta \in H_{loc}(\mathrm{curl}; \mathbb{R}^3)$ to (1.14) verify*

$$\int_{D_\delta \setminus \overline{D}_{\delta/2}} \sigma_l \mathbf{E}_\delta \cdot \overline{\mathbf{E}_\delta}\, \mathrm{d}\mathbf{x} + \int_{D_{\delta/2}} \sigma_a \mathbf{E}_\delta \cdot \overline{\mathbf{E}_\delta}\, \mathrm{d}\mathbf{x}$$
$$= \int_{\partial B_R} (\nu \times \overline{\mathbf{E}_\delta^+}) \cdot (\nu \times \nu \times \mathbf{H}_\delta^+)\, \mathrm{d}\sigma_x + \Re \int_{\partial D_\delta} (\nu \times \overline{\mathbf{E}^i}) \cdot (\nu \times \nu \times \mathbf{H}_\delta^+)\Big|_+ \mathrm{d}\sigma_x$$
$$+ \Re \int_{\partial D_\delta} (\nu \times \overline{\mathbf{E}_\delta^+})\Big|_+ \cdot (\nu \times \nu \times \mathbf{H}^i)\, \mathrm{d}\sigma_x. \tag{3.40}$$

Remark 3.3. It is remarked that the last two terms in the RHS of (3.40) in [7] were

$$\Re \int_{\partial B_R} (\nu \times \overline{\mathbf{E}^i}) \cdot (\nu \times \nu \times \mathbf{H}_\delta^+) \mathrm{d}\sigma_x + \Re \int_{\partial B_R} (\nu \times \overline{\mathbf{E}_\delta^+}) \cdot (\nu \times \nu \times \mathbf{H}^i)\, \mathrm{d}\sigma_x.$$

Here, we modify the two terms for the convenience of the present study.

Define a boundary operator Λ such that

$$\Lambda(\nu \times \mathbf{E}_\delta|_{\partial B_R}) = \nu \times \mathbf{H}_\delta|_{\partial B_R} : \quad \mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R) \to \mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R). \tag{3.41}$$

It is well-known that Λ is a bounded operator in $\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)$ (cf. [10,31]). By using Lemma 3.6, one can show:

Lemma 3.7. *Let* $\mathbf{E}_\delta$ *and* $\mathbf{H}_\delta$ *be solutions to the system* (1.14)*, where* D_δ *is the virtual domain described at the beginning of this section, then we have*

$$\int_{D_\delta^f\setminus\overline{D}_{\delta/2}} \|\mathbf{E}_\delta\|^2\,\mathrm{d}\mathbf{x} \leqq C\delta^{1-t}\Big(\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)} + \delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}\Big), \quad (3.42)$$

and

$$\int_{D_\delta^c\setminus\overline{D}_{\delta/2}} \|\mathbf{E}_\delta\|^2\,d\mathbf{x} \leqq C\delta^{2-t}\Big(\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)} + \delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}\Big),\ c\in\{a,b\}, \quad (3.43)$$

where the constant C *depends only on* D*,* ω *and* R*.*

Proof. First, by (3.40) and the fact that Λ is bounded in $\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)$, there holds

$$\int_{D_\delta\setminus\overline{D}_{\delta/2}} \sigma_l\mathbf{E}_\delta\cdot\overline{\mathbf{E}}_\delta\,d\mathbf{x} \leqq C\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)} + \mathbb{R}_1 + \mathbb{R}_2,$$

where by using (2.8) and (3.23) one further has

$$\begin{aligned}\mathbb{R}_1 &= \left|\Re\int_{\partial D_\delta}(\nu\times\overline{\mathbf{E}^i})\cdot(\nu\times\nu\times\mathbf{H}_\delta^+)\Big|_+\mathrm{d}\sigma_x\right| = \left|\Re\int_{\partial D_\delta}\overline{\mathbf{E}^i}\cdot(\nu\times\mathbf{H}_\delta^+)\Big|_+\mathrm{d}\sigma_x\right|\\ &\leqq \frac{\delta}{\omega}\left|\int_{S^a\cup S^b}\overline{\mathbf{E}^i}(\mathbf{z}_{\tilde{\mathbf{x}}})\cdot\mathcal{A}_{\delta,c}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}\right| + C\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)},\end{aligned}$$

and by using (3.21),

$$\begin{aligned}\mathbb{R}_2 &= \left|\Re\int_{\partial D_\delta}(\nu\times\overline{\mathbf{E}_\delta^+})\Big|_+\cdot(\nu\times\nu\times\mathbf{H}^i)\,\mathrm{d}\sigma_x\right| = \left|\Re\int_{\partial D_\delta}\overline{\mathbf{E}_\delta^+}\Big|_+\cdot(\nu\times\mathbf{H}^i)\,\mathrm{d}\sigma_x\right|\\ &\leqq C\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}.\end{aligned}$$

Combining those with (3.9), (3.11), together with the use of the definition of σ_l in (3.7), we can complete the proof. □

It is remarked that in the estimate (3.43), the dependence on the artificial R of the generic constant C can obviously be absorbed into the dependence on D.

In what follows, we shall estimate $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)}$ and $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^a\cup S^b)}$ separately. Clearly, it suffices to estimate $\|\tilde{\Phi}\|_{H^{-1/2}(S^f)^3}$ and $\|\tilde{\Phi}\|_{H^{-1/2}(S^a\cup S^b)^3}$, respectively. We recall that the $H^{-1/2}(\partial D)^3$-norm of the function $\tilde{\Phi}(\tilde{\mathbf{x}})$ is defined as follows:

$$\|\tilde{\Phi}\|_{H^{-1/2}(\partial D)^3} = \sup_{\|\varphi\|_{H^{1/2}(\partial D)^3}\leqq 1}\left|\int_{\partial D}\tilde{\Phi}(\tilde{\mathbf{x}})\cdot\overline{\varphi}(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}\right|. \quad (3.44)$$

We refer to [1,26,35] for more relevant discussions on the Sobolev spaces. To proceed, we first present the following important auxiliary Sobolev extension result:

Lemma 3.8. (Lemma 3.4 in [7]) *Suppose D is a simply connected domain with a $C^{1,1}$-smooth boundary ∂D. Then for any $\psi \in H^{1/2}(\partial D)^3$, there exists $\mathbf{W} \in H^2(D)^3$ such that*

$$\begin{aligned}
&\nu \times \mathbf{W} = 0 \quad on \quad \partial D,\\
&\nu \times (\nu \times (\nabla \times \mathbf{W})) = \nu \times (\nu \times \psi) \quad on \quad \partial D,\\
&\|\mathbf{W}\|_{H^2(D)^3} \leqq C\|\psi\|_{H^{1/2}(\partial D)^3},\\
&\mathbf{W} = 0 \quad in \quad D_{1/2},
\end{aligned}$$

where C is a constant depending only on D.

We proceed with the proof of Theorem 3.1.

Lemma 3.9. *Let $\tilde{\Phi}$ be defined in (3.28), where $(\mathbf{E}_\delta, \mathbf{H}_\delta)$ are the solutions to (1.14) with the corresponding ε_δ, μ_δ and σ_δ given by (3.7). Then there hold*

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} \leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash D_{\delta/2})^3}, \tag{3.45}$$

and

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^c)} \leqq C\delta^{-3/2+\beta'}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash D_{\delta/2})^3}, \quad c \in \{a, b\}, \tag{3.46}$$

where $\beta = \min\{1, -1+r+s, -1+t+s\}$ and $\beta' = \min\{1, -2+r+s, -2+t+s\}$.

Proof. It suffices to show that the same estimates in (3.45) and (3.46) hold for $\|\tilde{\Phi}(\tilde{\mathbf{x}})\|_{H^{-1/2}(S^c)^3}$, $c \in \{f, a, b\}$. First, we let $D_{1,\delta}$ be the region defined in Lemma 3.3. For any test function $\psi \in H^{1/2}(\partial D_{1,\delta})^3$, we introduce an auxiliary function $\mathbf{W} \in H^2(D_{1,\delta})^3$ which fulfils the conditions in Lemma 3.8. Let $\Phi_1(\tilde{\mathbf{x}})$ be defined by

$$\Phi_1(\tilde{\mathbf{x}}) := \begin{cases} \Phi(\tilde{\mathbf{x}}), & \tilde{\mathbf{x}} \in S^f,\\ \nu \times \tilde{\mathbf{E}}_\delta, & \tilde{\mathbf{x}} \in \partial D_{1,\delta}\backslash\overline{S^f}. \end{cases}$$

It then follows from the properties of $\mathbf{W}$ that

$$\begin{aligned}
\int_{\partial D_{1,\delta}} \tilde{\Phi}_1(\tilde{\mathbf{x}}) \cdot \psi(\tilde{\mathbf{x}})\, \mathrm{d}\sigma_{\tilde{x}} &= -\int_{\partial D_{1,\delta}} (\nu \times \tilde{\mathbf{E}}_\delta) \cdot (\nu \times (\nu \times \psi))\, \mathrm{d}\sigma_{\tilde{x}}\\
&= -\int_{\partial D_{1,\delta}} \tilde{\mathbf{E}}_\delta \cdot (\nu \times (\nabla \times \mathbf{W}))\, \mathrm{d}\sigma_{\tilde{x}}.
\end{aligned}$$

From its construction, it is readily seen that $\nu \times \mathbf{W}|_{\partial D_{1,\delta}} = 0$. Define

$$\widehat{\mathrm{Curl}\,\mathbf{W}} := (\tilde{\mathbf{B}}^T)^{-1}(\nabla \times \mathbf{W}), \quad \widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta} := (\tilde{\mathbf{B}}^T)^{-1}(\nabla \times \hat{\mathbf{E}}_\delta). \tag{3.47}$$

By using (3.12), (3.16), (3.19) and integration by parts, one has

$$
\begin{aligned}
\int_{\partial D_{1,\delta}} \tilde{\Phi}(\tilde{\mathbf{x}}) \cdot \psi(\tilde{\mathbf{x}})\, d\sigma_{\tilde{x}} &= -\delta^{-1} \int_{\partial D_{1,\delta}} \hat{\mathbf{E}}_\delta \cdot \left(\nu \times \widehat{\mathrm{Curl}\,\mathbf{W}}\right) d\sigma_{\tilde{x}} \\
&= -\delta^{-1}\left(\int_{\partial D_{1,\delta}} \hat{\mathbf{E}}_\delta \cdot \left(\nu \times \widehat{\mathrm{Curl}\,\mathbf{W}}\right) d\sigma_{\tilde{x}} - \int_{\partial D_{1,\delta}} \mathbf{W} \cdot \left(\nu \times \widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta}\right) d\sigma_{\tilde{x}}\right) \\
&= \delta^{-1}\left(\int_{D_{1,\delta}} \mathbf{W} \cdot \left(\nabla \times \widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta}\right) d\tilde{\mathbf{x}} - \int_{D_{1,\delta}} \hat{\mathbf{E}}_\delta \cdot \left(\nabla \times \widehat{\mathrm{Curl}\,\mathbf{W}}\right) d\tilde{\mathbf{x}}\right) \\
&= \delta^{-1} \int_{D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}}} \mathbf{W} \cdot \left(\nabla \times \widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta}\right) d\tilde{\mathbf{x}} - \delta \int_{D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}}} \tilde{\mathbf{E}}_\delta \cdot \left(\nabla \times (\nabla \times \mathbf{W})\right) d\tilde{\mathbf{x}}.
\end{aligned} \tag{3.48}
$$

For $\mathbf{x} \in D_\delta \setminus D_{\delta/2}$, there holds

$$
\nabla_{\mathbf{x}} \times \mathbf{E}_\delta = i\omega\mu_l \mathbf{H}_\delta, \quad \nabla_{\mathbf{x}} \times \mathbf{H}_\delta = -i\omega\left(\varepsilon_l + i\frac{\sigma_l}{\omega}\right)\mathbf{E}_\delta. \tag{3.49}
$$

By using change of variables and (3.16) in Lemma 3.2, one can derive from (3.49) that

$$
|\tilde{\mathbf{B}}|\tilde{\mathbf{B}}^{-1}\nabla_{\tilde{\mathbf{x}}} \times \hat{\mathbf{E}}_\delta = i\omega\tilde{\mu}_l \tilde{\mathbf{H}}_\delta, \quad |\tilde{\mathbf{B}}|\tilde{\mathbf{B}}^{-1}\nabla_{\tilde{\mathbf{x}}} \times \hat{\mathbf{H}}_\delta = -i\omega\left(\tilde{\varepsilon}_l + i\frac{\tilde{\sigma}_l}{\omega}\right)\tilde{\mathbf{E}}_\delta,
$$

which holds in $D\setminus\overline{D}_{1/2}$. By combining (3.7), (3.15) and (3.47), one can further show that in $D\setminus\overline{D}_{1/2}$,

$$
\nabla \times \widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta} = \omega^2\left(\delta^{r+s} + i\delta^{t+s}\frac{1}{\omega}\right)\tilde{\mathbf{E}}_\delta. \tag{3.50}
$$

Thus by plugging (3.50) into (3.48) one has

$$
\begin{aligned}
\int_{\partial D_{1,\delta}} \tilde{\Phi}(\tilde{\mathbf{x}}) \cdot \psi(\tilde{\mathbf{x}})\, d\sigma_{\tilde{x}} =& \omega^2(\delta^{-1+r+s} + i\delta^{-1+t+s}\frac{1}{\omega}) \int_{D_{1,\delta}\setminus\overline{D^f_{1/2,\delta/2}}} \mathbf{W} \cdot \tilde{\mathbf{E}}_\delta\, d\tilde{\mathbf{x}} \\
&- \delta \int_{D_{1,\delta}\setminus\overline{D^f_{1/2,\delta/2}}} \tilde{\mathbf{E}}_\delta \cdot \left(\nabla \times (\nabla \times \mathbf{W})\right) d\tilde{\mathbf{x}},
\end{aligned}
$$

which in combination with the fact that

$$
\|\tilde{\mathbf{E}}_\delta\|_{L^2(D_{1,\delta}\setminus\overline{D^f_{1/2,\delta/2}})^3} = \delta^{-1}\|\mathbf{E}_\delta\|_{L^2(D_\delta\setminus\overline{D_{\delta/2}})^3},
$$

immediately yields that

$$
\begin{aligned}
\left|\int_{\partial D_{1,\delta}} \tilde{\Phi}(\tilde{\mathbf{x}}) \cdot \psi(\tilde{\mathbf{x}})\, d\sigma_{\tilde{x}}\right| &\leqq C\delta^\beta \|\tilde{\mathbf{E}}_\delta\|_{L^2(D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}})^3} \|\mathbf{W}\|_{H^2(D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}})^3} \\
&\leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(D_\delta\setminus\overline{D_{\delta/2}})^3} \|\mathbf{W}\|_{H^2(D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}})^3},
\end{aligned} \tag{3.51}
$$

where $\beta = \min\{1, -1+r+s, -1+t+s\}$. By using Lemma 3.8, the definition (3.44) and the fact that

$$\|\mathbf{W}\|_{H^2(D_{1,\delta}\setminus\overline{D_{1/2,\delta/2}})^3} \leqq C\|\psi\|_{H^{1/2}(\partial D_{1,\delta})^3},$$

one readily has from (3.51) that

$$\|\tilde{\Phi}\|_{H^{-1/2}(S^f)^3} \leqq \|\tilde{\Phi}\|_{H^{-1/2}(\partial D_{1,\delta})^3} \leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(D_\delta\setminus\overline{D_{\delta/2}})^3},$$

which proves (3.45).

Next, we proceed to the proof of (3.46). In the sequel, we let $\mathbb{B}_r^a$ be a ball of radius $r \in \mathbb{R}_+$ centred at P_0 and $\mathbb{B}_r^b$ be a ball of radius r and centred at Q_0. For $c \in \{a, b\}$, we set $\mathbb{S}_r^c = \partial\mathbb{B}_r^c$. In what follows, we drop the dependence on r if $r = 1$. Moreover, in order to simplify the exposition and without loss of generality, we assume that $\mathbb{B}^c \subset \overline{D}$. Then by using exactly the same strategy as in the first part of the current proof, with $D_{1,\delta}$ replaced by the ball $\mathbb{B}^c$, along with the use of (3.20), one can show that for $c \in \{a, b\}$,

$$\begin{aligned}&\int_{\mathbb{S}^c} \tilde{\Phi}(\tilde{\mathbf{x}})\cdot\psi(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}\\ &= \delta^{-2}\int_{\mathbb{B}^c\setminus\overline{\mathbb{B}^c_{1/2}}} \mathbf{W}\cdot\Big(\nabla\times\widehat{\mathrm{Curl}\,\hat{\mathbf{E}}_\delta}\Big)\,\mathrm{d}\tilde{\mathbf{x}} - \delta\int_{\mathbb{B}^c\setminus\overline{\mathbb{B}^c_{1/2}}} \tilde{\mathbf{E}}_\delta\cdot\Big(\nabla\times(\nabla\times\mathbf{W})\Big)\mathrm{d}\tilde{\mathbf{x}},\end{aligned}$$

which together with (3.50) and the fact

$$\|\tilde{\mathbf{E}}_\delta\|_{L^2(\mathbb{B}^c\setminus\overline{\mathbb{B}^c_{1/2}})^3} \leqq \delta^{-3/2}\|\mathbf{E}_\delta\|_{L^2(D_\delta\setminus\overline{D_{\delta/2}})^3}, \quad c \in \{a, b\},$$

readily yields that

$$\left|\int_{\mathbb{S}^c} \tilde{\Phi}(\tilde{\mathbf{x}})\cdot\psi(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}\right| \leqq C\delta^{-3/2+\beta'}\|\mathbf{E}_\delta\|_{L^2(D_\delta\setminus\overline{D_{\delta/2}})^3}\|\mathbf{W}\|_{H^2(\mathbb{B}^c\setminus\overline{\mathbb{B}^c_{1/2}})^3},$$

where $\beta' = \min\{1, -2+r+s, -2+t+s\}$. By using the fact that

$$\|\mathbf{W}\|_{H^2(\mathbb{B}^c\setminus\overline{\mathbb{B}^c_{1/2}})^3} \leqq C\|\psi\|_{H^{1/2}(\mathbb{S}^c)^3}, \quad c = \{a, b\},$$

one readily has

$$\|\tilde{\Phi}\|_{H^{-1/2}(S^c)^3} \leqq \|\tilde{\Phi}\|_{H^{-1/2}(\mathbb{S}^c)^3} \leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(L^2(D_\delta\setminus\overline{D_{\delta/2}})^3},$$

which proves (3.45).

The proof is complete. □

Proof of Theorem 3.1. It is straightforward to see from (3.39) that we only need to derive the estimates of $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)}$ and $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}$. To that end, we have by using (2.7), (3.13), (3.29) and (3.35) that

$$\begin{aligned}\|\nu\times\mathbf{E}_\delta^+\|_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)} &\leqq C\|\nu\times\nabla\times\mathcal{S}_{D_\delta}[\mathbf{a}]\|_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)}\\ &\leqq C\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(S^f)} + C\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}\\ &\leqq C\delta\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + C\delta^2(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1). \quad (3.52)\end{aligned}$$

Next, by combining (3.52), (3.35), (3.42), (3.43), (3.45) and (3.46), together with the use of the facts that both Λ and Λ^- are bounded, and $\beta' \leqq \beta$ (see Lemma 3.9), one can further deduce that

$$\begin{aligned}\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} &\leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash D_{\delta/2})^3} + C\delta^{-3/2+\beta'}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash D_{\delta/2})^3}\\ &\leqq C\delta^{-3/2+\beta'+1/2-t/2}\Big(\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)}\\ &\quad + \mathcal{O}(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})\Big)^{1/2}\\ &\leqq C\delta^{\beta'-t/2-1}(\delta\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + \delta^2\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + \delta^2)\\ &\quad + C\delta^{\beta'-t/2-1/2}(\|\tilde{\Phi}\|^{1/2}_{\mathrm{TH}^{-1/2}(\partial D)} + 1),\end{aligned}$$

which in turn yields (noting that $\beta' - t/2 \geqq 1/2$)

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} \leqq C\delta^{\beta'-t/2-1/2}(\|\tilde{\Phi}\|^{1/2}_{\mathrm{TH}^{-1/2}(\partial D)} + 1).$$

Hence one has

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} \leqq C\delta^{\beta'-t/2-1/2}. \tag{3.53}$$

It is remarked that in (3.53), the generic constant obviously depends on R, but as was remarked earlier that such a dependence can be absorbed into the dependence on D. Next, by using (3.45) and (3.53), there holds

$$\begin{aligned}\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} &\leqq C\delta^{-1+\beta}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash D_{\delta/2})^3}\\ &\leqq C\delta^{-1+\beta+1/2-t/2}\Big(\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}_{\mathrm{div}}^{-1/2}(\partial B_R)} + \mathcal{O}(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)})\Big)^{1/2}\\ &\leqq C\delta^{-1/2+\beta-t/2}(\delta\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} + \delta^2) + C\delta^{\beta-t/2},\end{aligned}$$

and thus

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^f)} \leqq C\delta^{\beta-t/2}. \tag{3.54}$$

By plugging (3.53) and (3.54) into (3.39), one finally has (3.8).

The proof is complete.

4. Regularized Partial-cloaking of EM Waves

In this section, we consider the regularized partial-cloaking of EM waves by taking the generating set Γ_0 to be a flat subset on a plane in $\mathbb{R}^3$. In order to ease our exposition, we stick our subsequent study to a specific example considered in [9,25] for the partial-cloaking of acoustic waves, where Γ_0 is taken to be a square on $\mathbb{P}_2 := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$. The virtual domain is constructed as follows; see Fig. 2 for a schematic illustration. Let $\mathbf{n} \in \mathbb{S}^2$ be the unit normal vector to Γ_0. Let

$$D_q^0 := \Gamma_0(\mathbf{x}) \times [\mathbf{x} - \tau\cdot\mathbf{n}, \mathbf{x} + \tau\cdot\mathbf{n}], \quad \mathbf{x} \in \overline{\Gamma}_0, \quad 0 \leqq \tau \leqq q, \tag{4.1}$$

where we identify Γ_0 with its parametric representation $\Gamma_0(\mathbf{x})$. We denote by D_q^1 the union of the four side half-cylinders and D_q^2 the union of the four corner quarter-balls in Fig. 2. Let

$$D_q := D_q^0 \cup D_q^1 \cup D_q^2. \tag{4.2}$$

Set S_q^1 and S_q^2 to denote

$$S_q^1 := \partial D_q \cap \partial D_q^1, \quad S_q^2 := \partial D_q \cap \partial D_q^2,$$

and

$$D_q^f := D_q^1 \cup D_q^2.$$

The upper and lower-surfaces of D_q are respectively denoted by

$$\Gamma_q^1 := \{\mathbf{x} + q \cdot \mathbf{n};\ \mathbf{x} \in \Gamma_0\} \quad \text{and} \quad \Gamma_q^2 := \{\mathbf{x} - q \cdot \mathbf{n};\ \mathbf{x} \in \Gamma_0\}.$$

Define $S_q^0 := \Gamma_q^1 \cup \Gamma_q^2$. We then have $\partial D_q = S_q^0 \cup S_q^1 \cup S_q^2$. Similar to our notations in Section 3, we let $\delta \in \mathbb{R}_+$ denote the asymptotically small regularization parameter and D_δ be the virtual domain. Clearly, we have

$$\partial D_\delta = S_\delta^0 \cup S_\delta^1 \cup S_\delta^2. \tag{4.3}$$

In what follows, if $q \equiv 1$, we drop the dependence on q of D_q, S_q^0, S_q^1 and S_q^2, and simply write them as D, S^0, S^1, and S^2. In concluding the description of the virtual domain for the partial-cloaking construction, we would like to emphasize that our subsequent study can be easily extended to a much more general case where the generating set Γ_0 can be a bounded subset on a plane with a convex and piecewise smooth boundary (in the topology induced from the plane).

We introduce a blowup transformation A which maps D_δ to D exactly as that in [25]. We stress that in D_δ^0 the blow-up transformation takes the following form

$$A(\mathbf{y}) = \tilde{\mathbf{y}} := \left(\frac{\mathbf{e}_3\mathbf{e}_3^T}{\delta} + \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T \right)\mathbf{y}, \quad \mathbf{y} \in D_\delta^0.$$

where $\mathbf{y} \in D_\delta^0$ and $\tilde{\mathbf{y}} \in D^0$, and the three Euclidean unit vectors are as follows

$$\mathbf{e}_1 = (1, 0, 0)^T, \quad \mathbf{e}_2 = (0, 1, 0)^T, \quad \mathbf{e}_3 = (0, 0, 1)^T.$$

We are now in position to introduce the lossy layer for our partial-cloaking device:

$$\begin{aligned} &\varepsilon_\delta(\mathbf{x}) = \varepsilon_l(\mathbf{x}) := \delta^r |\mathbf{B}|\mathbf{B}^{-1}, \quad \mu_\delta(\mathbf{x}) = \mu_l(\mathbf{x}) := \delta^s |\mathbf{B}|\mathbf{B}^{-1}, \\ &\sigma_\delta(\mathbf{x}) = \sigma_l(\mathbf{x}) := \delta^t |\mathbf{B}|\mathbf{B}^{-1}, \quad \text{for} \quad \mathbf{x} \in D_\delta \backslash \overline{D}_{\delta/2}, \end{aligned} \tag{4.4}$$

where $\mathbf{B}(\mathbf{x}) := \nabla_{\mathbf{x}} A(\mathbf{x})$ is the Jacobian matrix of the blowup transformation A. It is obvious that

$$\mathbf{B} = \frac{\mathbf{e}_3\mathbf{e}_3^T}{\delta} + \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T \quad \text{in} \quad D_\delta^0, \tag{4.5}$$

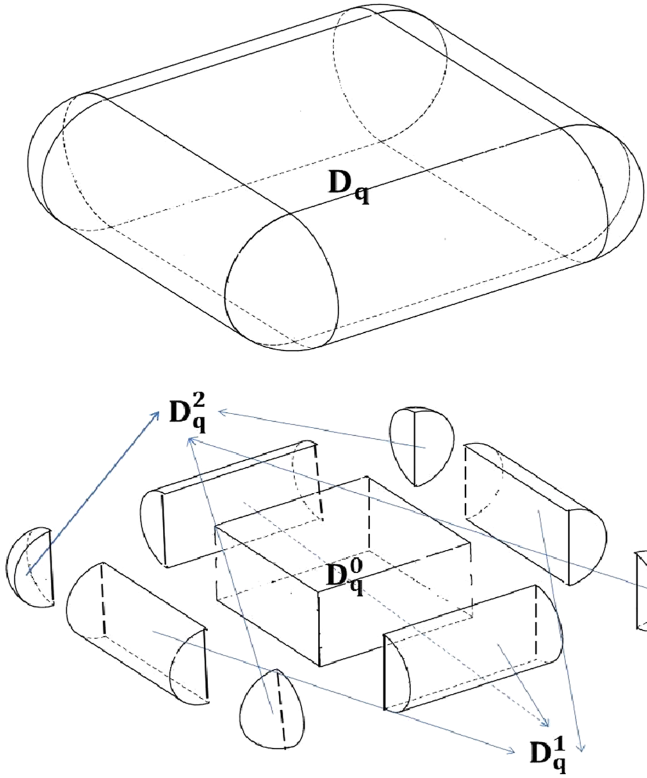

Fig. 2. Schematic illustration of the domain D_q for the regularized partial-cloak

is a constant matrix and

$$\mathbf{Bn} = \frac{\mathbf{n}}{\delta}, \quad \mathbf{Be}_1 = \mathbf{e}_1, \quad \mathbf{Be}_2 = \mathbf{e}_2 \quad \text{in} \quad D_\delta^0. \tag{4.6}$$

Thus one has

$$|\mathbf{B}| = 1/\delta, \quad \text{in} \quad D_\delta^0.$$

Furthermore, for $\mathbf{x} \in D_\delta$, one can show by direct calculations that (3.10) and (3.11) are also valid for the $\mathbf{B}$ defined above.

We are now in a position to present the main theorem of this section in quantifying our partial-cloaking construction.

Theorem 4.1. *Let D_δ be defined in (4.2) with its boundary given by (4.3). Let $(\mathbf{E}_\delta, \mathbf{H}_\delta)$ be the pair of solutions to (1.14) with $\{\Omega; \varepsilon_\delta, \mu_\delta, \sigma_\delta\} \subset \{\mathbb{R}^3; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ defined in (1.3) and $\{D_\delta \backslash \overline{D}_{\delta/2}; \varepsilon_\delta, \mu_\delta, \sigma_\delta\}$ given in (4.4). Let $\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})$ be the scattering amplitude of $\mathbf{E}_\delta$. Define*

$$\beta_j = \min\{1, -j + r + s, -j + t + s\}, \quad j = 0, 1, 2,$$

and for $\varepsilon \in \mathbb{R}_+$ with $\varepsilon \ll 1$,

$$\Sigma_p := \{\mathbf{p} \in \mathbb{S}^2;\ \|\mathbf{p} \times \mathbf{n}\| \leqq \varepsilon\}, \quad \Sigma_d := \{\mathbf{d} \in \mathbb{S}^2;\ |\mathbf{d} \cdot \mathbf{n}| \leqq \varepsilon\}. \tag{4.7}$$

If r, s and t are chosen such that $\beta_2 - t/2 \geqq 3/2$, then there exists $\delta_0 \in \mathbb{R}_+$ such that when $\delta < \delta_0$,

$$\|\mathbf{A}_\infty^\delta(\hat{\mathbf{x}}; \mathbf{p}, \mathbf{d})\| \leqq C(\varepsilon + \delta), \quad \mathbf{p} \in \Sigma_p, \quad \mathbf{d} \in \Sigma_d, \quad \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{4.8}$$

where C is a positive constant depending on ω and D, but independent of ε_a, μ_a, σ_a and $\hat{\mathbf{x}}$, $\mathbf{p}$, $\mathbf{d}$.

Remark 4.1. Similar to Remark 3.1, one readily has from Theorems 1.1 and 4.1 that the push-forwarded structure in (1.2) produces an approximate partial-cloaking device which is capable of nearly cloaking an arbitrary EM content. As an illustrative example, one can take $r = 0$, $s = 7/2$ and $t = -1$ in Theorem 4.1 to achieve an approximate partial-cloak.

Remark 4.2. Since $\mathbf{p} \cdot \mathbf{d} = 0$ (cf. (1.7)), one has

$$(\mathbf{d} \cdot \mathbf{n})\mathbf{p} = \mathbf{d} \times (\mathbf{p} \times \mathbf{n}), \tag{4.9}$$

from which one can infer that the definition of Σ_d in (4.7) is redundant. However, we specified it for clarity and definiteness.

4.1. Auxiliary Lemmas and Asymptotic Expansions

We shall follow a similar strategy of the proof for Theorem 3.1 in proving Theorem 4.1. In what follows, we adopt similar notations as those in Section 3. If we let $\mathbf{z}$ denote the space variable on Γ_0, then for any $\mathbf{y} \in \partial D_q$, we define $\mathbf{z}_y$ to be the projection of $\mathbf{y}$ onto Γ_0. We also define $\tilde{\mathbf{a}}(\tilde{\mathbf{y}}) := \mathbf{a}(\mathbf{y})$ and $\tilde{\mathbf{y}} := A(\mathbf{y}) \in \overline{D}$ for $\mathbf{y} \in \overline{D_\delta}$.

The following lemma is a counterpart to Lemma 3.4 in Section 3.

Lemma 4.1. *Let D_δ be described in (4.2) and (4.3). Let $\iota_\delta(\mathbf{x})$, $\iota_{1,\delta}(\tilde{\mathbf{x}})$ be two regions defined similarly as those in Lemma 3.4. Set*

$$\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) := \tilde{\mathbf{x}} - \tilde{\mathbf{y}} + (1/\delta - 1)(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}).$$

Define

$$\mathcal{M}^\omega_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) := \text{p.v.} \quad \nu_{\tilde{\mathbf{x}}} \times \nabla_{\mathbf{z}_{\tilde{x}}} \times \int_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} G_\omega(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, d\sigma_{\tilde{y}}, \tag{4.10}$$

and

$$\mathcal{M}_{\delta,2}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) := \text{p.v.} \quad -\frac{1}{4\pi}\nu_{\tilde{\mathbf{x}}} \times \int_{S^2\cap\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} \frac{\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})}{\|\kappa(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})\|^3} \times \tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, d\sigma_{\tilde{y}}.$$

Define

$$\mathcal{L}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) := \nu_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \nabla_{\mathbf{x}} \times \mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}),$$

$$\mathcal{L}_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}}^{\omega}[\tilde{\mathbf{a}}](\mathbf{x}) := \nu_{\tilde{\mathbf{x}}} \times \nabla_{\mathbf{z}_{\tilde{x}}} \times \nabla_{\mathbf{z}_{\tilde{x}}} \times \int_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}} G_\omega(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, \mathrm{d}\sigma_{\tilde{y}},$$

$$\mathcal{A}_{\delta,2}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) := -\frac{1}{4\pi}\nu_{\tilde{\mathbf{x}}} \times \int_{S^2\cap\iota_{1,\delta}(\tilde{\mathbf{x}})} \frac{1}{\|\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\|^3}\left(\frac{\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})^T}{\|\kappa(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\|^2} - I\right)\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, \mathrm{d}\sigma_{\tilde{y}}.$$

Then one has

$$\mathcal{M}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) = \mathcal{M}_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}}^{\omega}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathcal{M}_{\delta,2}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathcal{O}(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}), \quad (4.11)$$

and

$$\mathcal{L}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) = \mathcal{L}_{S^0\backslash\overline{\iota_{1,\delta}(\tilde{\mathbf{x}})}}^{\omega}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \frac{1}{\delta}\mathcal{A}_{\delta,2}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}}) + \mathcal{O}(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}). \quad (4.12)$$

Proof. By following a similar argument to that in the proof of Lemma 3.4, one can compute for $\mathbf{y} \in (S_\delta^0 \cup S_\delta^1)\backslash\overline{\iota_\delta(\mathbf{x})}$ that

$$\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} - \mathbf{z}_x - (\mathbf{y} - \mathbf{z}_y) + \mathbf{z}_x - \mathbf{z}_y\| = \|\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}\| + \mathcal{O}(\delta),$$

and

$$\langle \mathbf{x} - \mathbf{y}, \nu_{\mathbf{x}}\rangle = \langle \mathbf{z}_x - \mathbf{z}_y, \nu_{\tilde{\mathbf{x}}}\rangle + \mathcal{O}(\delta), \quad e^{i\omega|\mathbf{x}-\mathbf{y}|} = e^{i\omega|\mathbf{z}_{\tilde{x}}-\mathbf{z}_{\tilde{y}}|} + \mathcal{O}(\delta).$$

One also notes that for $\mathbf{y} \in S_\delta^2 \cap \iota_\delta(\mathbf{x})$

$$\|\mathbf{x} - \mathbf{y}\| = \delta\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|, \quad e^{i\omega\|\mathbf{x}-\mathbf{y}\|} = 1 + \mathcal{O}(\delta).$$

By using the above facts, the proof can then be completed by using the similar expansion method as that in the proof of Lemma 3.4. □

Next, by straightforward calculations, one can show that for D_δ described in (4.2) and (4.3), there holds the following far-field expansion for $\mathbf{x} \in \mathbb{R}^3\backslash\overline{D}_\delta$,

$$\begin{aligned}\mathcal{S}_{D_\delta}^{\omega}[\mathbf{a}](\mathbf{x}) &= \int_{\partial D_\delta} G_\omega(\mathbf{x} - \mathbf{y})\mathbf{a}(\mathbf{y})d\sigma_y = \int_{S^0} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\mathrm{d}\sigma_{\tilde{y}}\\ &+ \delta\int_{S^1} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\,\mathrm{d}\sigma_{\tilde{y}} + \delta\int_{S^0}(\tilde{\mathbf{y}} - \mathbf{z}_{\tilde{y}})\cdot\nabla G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\mathrm{d}\sigma_{\tilde{y}}\\ &+ \mathcal{O}\Big(\delta^2\|\mathbf{x}\|^{-1}\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}\Big). \quad (4.13)\end{aligned}$$

With the above expansion, we can show the following theorem.

Theorem 4.2. *Let* $\mathbf{E}_\delta$ *be the solution to* (1.14) *with* D_δ *described in* (4.2) *and* (4.3). *Define* $\tilde{\Phi}(\tilde{\mathbf{y}}) := \Phi(\mathbf{y}) = \nu_{\mathbf{y}} \times \mathbf{E}_\delta(\mathbf{y})\Big|_{\partial D_\delta}^{+}$, *then there holds for* $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$

$$(\mathbf{E}_\delta - \mathbf{E}^i)(\mathbf{x}) = \int_{\Gamma_0} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}}) \mathbb{M}[\mathbf{n} \times \mathbf{E}^i(\mathbf{z})](\mathbf{z}_{\tilde{y}})\, \mathrm{d}\sigma_{\mathbf{z}_{\tilde{y}}} \\ + \mathcal{O}\left(\|\mathbf{x}\|^{-1} \left(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1) \right) \right), \tag{4.14}$$

where

$$\mathbb{M} := \left(-\frac{I}{4} + \mathcal{M}_{\Gamma_0}^{\omega} \right)^{-1} \mathcal{M}_{\Gamma_0}^{\omega} \left(\frac{I}{4} + \mathcal{M}_{\Gamma_0}^{\omega} \right)^{-1} \tag{4.15}$$

with $\mathcal{M}_{\Gamma_0}^{\omega}$ *defined by*

$$\mathcal{M}_{\Gamma_0}^{\omega}[\Theta(\mathbf{z})](\mathbf{z}_{\tilde{x}}) := \mathbf{n} \times \nabla_{\mathbf{z}_{\tilde{x}}} \times \int_{\Gamma_0} G_\omega(\mathbf{z}_{\tilde{x}} - \mathbf{z}_{\tilde{y}}) \Theta(\mathbf{z}_{\tilde{y}})\, \mathrm{d}\sigma_{\mathbf{z}_{\tilde{y}}}. \tag{4.16}$$

Proof. We first recall that the solution to (1.14) can be represented by (2.7) and (2.8), where the density function **a** satisfies (2.9). Next, by using the fact that

$$-\frac{I}{2} + \mathcal{M}_{S^0}^{\omega} : \mathrm{TH}^{-1/2}(S^0) \to \mathrm{TH}^{-1/2}(S^0)$$

is invertible (as mentioned in the proof of Theorem 3.2, the invertibility can be shown by following a complete similar argument to the proof of Theorem 4.2 in [9], along with the use of Proposition 4.7 in [2]), and using (2.9), (4.11), together with the use of the expansion of $\mathbf{E}^i$ in $\mathbf{z}$, one has, by direct calculations, that

$$\begin{aligned} \tilde{\mathbf{a}}(\tilde{\mathbf{y}}) &= \left(-\frac{I}{2} + \mathcal{M}_{S^0}^{\omega} \right)^{-1} \left[\nu \times \mathbf{E}^i(\mathbf{z}) + \tilde{\Phi} \right](\tilde{\mathbf{y}}) + \mathcal{O}(\delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)) \\ &= \left(-\frac{I}{2} + \mathcal{M}_{S^0}^{\omega} \right)^{-1} \left[\nu \times \mathbf{E}^i(\mathbf{z}) \right](\tilde{\mathbf{y}}) \\ &\quad + \mathcal{O}\left(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1) \right) \end{aligned} \tag{4.17}$$

for all $\tilde{\mathbf{y}} \in S^0 \cup S^1$. Here, it is noted that for $\tilde{\mathbf{y}} \in \Gamma^1$, $(\tilde{\mathbf{y}} - 2\mathbf{n}) \in \Gamma^2$, we define $\tilde{\psi}(\tilde{\mathbf{y}}) := \tilde{\mathbf{a}}(\tilde{\mathbf{y}} - 2\mathbf{n})$ for $\tilde{\mathbf{y}} \in \Gamma^1$ and $\tilde{\psi}(\tilde{\mathbf{y}}) := \tilde{\mathbf{a}}(\tilde{\mathbf{y}} + 2\mathbf{n})$ for $\tilde{\mathbf{y}} \in \Gamma^2$. By using the fact that

$$\nu_{\tilde{\mathbf{y}} - 2n_{\tilde{\mathbf{y}}}} = -\nu_{\tilde{\mathbf{y}}} = -\mathbf{n} \quad \text{for} \quad \tilde{\mathbf{y}} \in \Gamma^1,$$

and the definition in (4.10), one obtains (assuming for a while that $\tilde{\mathbf{x}} \in S^0$)

$$\tilde{\mathbf{a}}(\tilde{\mathbf{y}} - 2\mathbf{n}) = \tilde{\psi}(\tilde{\mathbf{y}}) = \left(\frac{I}{2} + \mathcal{M}_{S^0}^{\omega} \right)^{-1} \left[\mathbf{n} \times \mathbf{E}^i(\mathbf{z}_{\tilde{x}}) \right](\tilde{\mathbf{y}}) \\ + \mathcal{O}\left(\|\mathbf{x}\|^{-1} \left(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1) \right) \right) \tag{4.18}$$

for all $\tilde{\mathbf{y}} \in \Gamma^1$. By inserting (4.17) and (4.18) into (4.13), one then has that for $\mathbf{x} \in \mathbb{R}^3 \backslash \overline{D}$,

$$\begin{aligned}\mathcal{S}^{\omega}_{D_\delta}[\mathbf{a}](\mathbf{x}) &= \int_{S^0} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\tilde{\mathbf{a}}(\tilde{\mathbf{y}})\, d\sigma_{\tilde{y}} + \mathcal{O}\left(\|\mathbf{x}\|^{-1}\left(\delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)\right)\right)\\ &= \int_{\Gamma^1} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\big(\tilde{\mathbf{a}}(\tilde{\mathbf{y}}) + \tilde{\psi}(\tilde{\mathbf{y}})\big)\, d\sigma_{\tilde{y}} + \mathcal{O}\left(\|\mathbf{x}\|^{-1}\left(\delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)\right)\right)\\ &= \int_{\Gamma^1} G_\omega(\mathbf{x} - \mathbf{z}_{\tilde{y}})\hat{\mathbb{M}}[\mathbf{n} \times \mathbf{E}^i(\mathbf{z})](\tilde{\mathbf{y}})\, d\sigma_{\tilde{y}}\\ &\quad + \mathcal{O}\left(\|\mathbf{x}\|^{-1}\left(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)\right)\right),\end{aligned}$$

where $\hat{\mathbb{M}}$ is defined by

$$\hat{\mathbb{M}} := 2\left(-\frac{I}{2} + \mathcal{M}^{\omega}_{S^0}\right)^{-1} \mathcal{M}^{\omega}_{S^0} \left(\frac{I}{2} + \mathcal{M}^{\omega}_{S^0}\right)^{-1}. \tag{4.19}$$

Since the function $\Theta(\mathbf{z}) := \mathbf{n} \times \mathbf{E}^i(\mathbf{z})$ depends only on $\mathbf{z} \in \Gamma_0$, and hence by the definition in (4.16), one readily verifies that

$$\mathcal{M}^{\omega}_{\Gamma_0}[\Theta](\mathbf{z}_{\tilde{x}}) = \frac{1}{2}\mathcal{M}^{\omega}_{S^0}[\hat{\Theta}(\tilde{\mathbf{y}})](\mathbf{z}_{\tilde{x}} + \mathbf{n}), \quad \tilde{\mathbf{x}} \in \Gamma_1,$$

for $\hat{\Theta}(\tilde{\mathbf{y}}) = \mathbf{n} \times \mathbf{E}^i(\mathbf{z}_{\tilde{y}})$, $\tilde{\mathbf{y}} \in S^0$. Therefore, we can replace the operator $\hat{\mathbb{M}}$ in (4.19) to be

$$\mathbb{M} := \left(-\frac{I}{4} + \mathcal{M}^{\omega}_{\Gamma_0}\right)^{-1} \mathcal{M}^{\omega}_{\Gamma_0} \left(\frac{I}{4} + \mathcal{M}^{\omega}_{\Gamma_0}\right)^{-1},$$

and the proof is complete. □

By using Theorem 4.2, we next consider a particular case by assuming that $\tilde{\Phi} \equiv 0$. Physically speaking, this corresponds to the case that D_δ is a so-called perfectly electric conductor. The result in the next theorem already partly reveals the partial-cloaking effect.

Theorem 4.3. *Let D_δ be as described in (4.2) and (4.3). Consider the following scattering problem*

$$\begin{cases}\nabla \times \mathbf{E}^p_\delta - i\omega \mathbf{H}^p_\delta = 0 & in \;\; \mathbb{R}^3 \backslash \overline{D}_\delta,\\ \nabla \times \mathbf{H}^p_\delta + i\omega \mathbf{E}^p_\delta = 0 & in \;\; \mathbb{R}^3 \backslash \overline{D}_\delta,\\ \nu \times \mathbf{E}^p_\delta\big|_+ = 0 & on \;\; \partial D_\delta,\end{cases} \tag{4.20}$$

subject to the Silver–Müller radiation condition:

$$\lim_{|\mathbf{x}| \to \infty} \|\mathbf{x}\|((\mathbf{H}^p_\delta - \mathbf{H}^i) \times \hat{\mathbf{x}} - (\mathbf{E}^p_\delta - \mathbf{E}^i)) = 0. \tag{4.21}$$

Let $\mathbf{E}_\infty^\delta(\hat{\mathbf{x}};\mathbf{E}^i)$ be the corresponding scattering amplitude to (4.20)–(4.21). Then there holds

$$\left\|\mathbf{E}_\infty^\delta(\hat{\mathbf{x}};\mathbf{E}^i)+\frac{1}{2\pi}\int_{\Gamma_0} e^{-i\omega\frac{4\pi}{3}\sum_{m=-1}^{1} Y_1^m(\hat{\mathbf{x}})\overline{Y_1^m(\hat{\mathbf{z}}_{\tilde{y}})}|\mathbf{z}_{\tilde{y}}|}\mathbb{M}[\mathbf{n}\times\mathbf{E}^i(\mathbf{z})](\mathbf{z}_{\tilde{y}})\,\mathrm{d}\sigma_{z_{\tilde{y}}}\right\|\leqq C\delta, \tag{4.22}$$

where $\mathbb{M}$ is defined in (4.15) and C depends only on ω and D. Furthermore, if there holds

$$\|\mathbf{n}\times\mathbf{p}\|\leqq\varepsilon,\quad \varepsilon\ll 1, \tag{4.23}$$

then for sufficient small $\delta\in\mathbb{R}_+$, one has

$$\|\mathbf{E}_\infty^\delta(\hat{\mathbf{x}};\mathbf{E}^i)\|\leqq C(\varepsilon+\delta), \tag{4.24}$$

where C depends only on ω and D.

Proof. We start with the following addition formula (see, example, [33]),

$$\frac{1}{\|\mathbf{x}-\mathbf{y}\|}=4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}Y_n^m(\xi,\vartheta)\overline{Y_n^m(\xi',\vartheta')}\frac{q'^n}{q^{n+1}}, \tag{4.25}$$

where (q,ξ,ϑ) and (q',ξ',ϑ') are the spherical coordinates of $\mathbf{x}$ and $\mathbf{y}$, respectively; and Y_n^m is the spherical harmonic function of degree n and order m. For simplicity, the parameters (q,ξ,ϑ) and (q',ξ',ϑ') shall be replaced by $(\|\mathbf{x}\|,\hat{\mathbf{x}})$ and $(\|\mathbf{y}\|,\hat{\mathbf{y}})$, respectively. It then follows by (4.25) that

$$\|\mathbf{x}-\mathbf{y}\|=\|\mathbf{x}\|\left(1-\frac{4\pi}{3}\sum_{m=-1}^{1}Y_1^m(\hat{\mathbf{x}})\overline{Y_1^m(\hat{\mathbf{y}})}\frac{\|\mathbf{y}\|}{\|\mathbf{x}\|}\right)+\mathcal{O}(\|\mathbf{x}\|^{-1}). \tag{4.26}$$

The solution $\mathbf{E}_\delta^p$ in (4.20) and (4.21) can be represented by

$$\mathbf{E}_\delta^p=\mathbf{E}^i+\mathcal{A}_{D_\delta}^{\omega}[\mathbf{a}]\quad\text{in}\quad\mathbb{R}^3\backslash D_\delta,$$

with $\mathbf{a}$ satisfying

$$\left(-\frac{I}{2}+\mathcal{M}_{D_\delta}^{\omega}\right)[\mathbf{a}](\mathbf{y})=-\nu_{\mathbf{y}}\times\mathbf{E}^i(\mathbf{y}),\quad \mathbf{y}\in\partial D_\delta.$$

Then one can easily derive from the expansion formula (4.14) with $\tilde{\Phi}=0$ that

$$(\mathbf{E}_\delta-\mathbf{E}^i)(\mathbf{x})=2\int_{\Gamma_0}G_\omega(\mathbf{x}-\mathbf{z}_{\tilde{y}})\mathbb{M}[\mathbf{n}\times\mathbf{E}^i(\mathbf{z})](\mathbf{z}_{\tilde{y}})\,\mathrm{d}\sigma_{z_{\tilde{y}}}+\mathcal{O}(\|\mathbf{x}\|^{-1}\delta),\ \mathbf{x}\in\mathbb{R}^3\backslash D_\delta, \tag{4.27}$$

where $\mathbb{M}$ is defined by (4.15). By combining (4.27) and (4.26), we then have

$$\mathbf{E}_\infty^\delta(\hat{\mathbf{x}};\mathbf{E}^i)=-\frac{1}{2\pi}\int_{\Gamma_0} e^{-i\omega\frac{4\pi}{3}\sum_{m=-1}^{1} Y_1^m(\hat{\mathbf{x}})\overline{Y_1^m(\hat{\mathbf{z}}_{\tilde{y}})}|\mathbf{z}_{\tilde{y}}|}\mathbb{M}[\mathbf{n}\times\mathbf{E}^i(\mathbf{z})](\mathbf{z}_{\tilde{y}})\,\mathrm{d}\sigma_{z_{\tilde{y}}}+\mathcal{O}(\delta), \tag{4.28}$$

which readily proves (4.22). Next by the definition of $\mathbb{M}$ in (4.15), one can show that

$$\|\mathbb{M}[\mathbf{n}\times\mathbf{E}^i(\mathbf{z})]\|_{\mathrm{TH}^{-1/2}(\Gamma_0)} \leqq C\|\mathbf{n}\times\mathbf{E}^i(\mathbf{z})\|_{\mathrm{TH}^{-1/2}(\Gamma_0)},$$

where C depends only on Γ_0 and ω. This together with (4.28) further implies that

$$\|\mathbf{E}_\infty^\delta(\hat{\mathbf{x}};\mathbf{E}^i)\| \leqq C(\|\mathbf{n}\times\mathbf{E}^i(\mathbf{z})\|_{\mathrm{TH}^{-1/2}(\Gamma_0)}+\delta). \tag{4.29}$$

Finally, by inserting the condition (4.23) into (4.29) and direct calculations, one can easily show (4.24).

The proof is complete. □

4.2. Proof of Theorem 4.1

In this section, we present the proof of Theorem 4.1, which follows a similar spirit to that of Theorems 3.1 and 4.3. The major idea is to control the norm $\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}$, which was taken to be identically zero in Theorem 4.3. Before this, we present two auxiliary lemmas.

Lemma 4.2. *Let D^j, $j=0,1,2$ be defined at the beginning of this section. Let $D^j_{1,\delta}$, $j=0,1$ be constructed similarly as those in Lemma 3.3. Let $\mathbf{V}$, $\mathbf{W}$ and $\hat{\mathbf{V}}$, $\hat{\mathbf{W}}$ be defined similarly as those in Lemma 3.2. Then one has*

$$\int_{\partial D^j_{1,\delta}}(\nu_{\tilde{\mathbf{x}}}\times\tilde{\mathbf{V}}(\tilde{\mathbf{x}}))\cdot\mathbf{W}(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}} = \delta^{-j}\int_{\partial D^j_{1,\delta}}(\nu_{\tilde{\mathbf{x}}}\times\hat{\mathbf{V}}(\tilde{\mathbf{x}}))\cdot\hat{\mathbf{W}}(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}},\quad j=0,1,$$
$$\int_{\partial D^2}(\nu_{\tilde{\mathbf{x}}}\times\tilde{\mathbf{V}}(\tilde{\mathbf{x}}))\cdot\mathbf{W}(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}} = \delta^{-2}\int_{\partial D^2}(\nu_{\tilde{\mathbf{x}}}\times\hat{\mathbf{V}}(\tilde{\mathbf{x}}))\cdot\hat{\mathbf{W}}(\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}.$$

Proof. The proof follows directly from (3.17), (3.18) by using change of variables in the corresponding integrals. □

Lemma 4.3. *Let $(\mathbf{E}_\delta,\mathbf{H}_\delta)$ be the pair of solutions to (1.14) with $\{\Omega;\varepsilon_\delta,\mu_\delta,\sigma_\delta\}\subset\{\mathbb{R}^3;\varepsilon_\delta,\mu_\delta,\sigma_\delta\}$ defined in (1.3) and $\{D_\delta\backslash\overline{D}_{\delta/2};\varepsilon_\delta,\mu_\delta,\sigma_\delta\}$ given in (4.4). Then there hold the following estimates for $j=0,1,2$,*

$$\int_{D^j_\delta\backslash\overline{D_{\delta/2}}}\|\mathbf{E}_\delta\|^2\mathrm{d}\mathbf{x} \leqq C\delta^{j-t}\Big(\|\nu\times\mathbf{E}_\delta^+\|^2_{\mathrm{TH}^{-1/2}_{\mathrm{div}}(\partial B_R)}+\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(S^0)}+\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}\Big),$$

where the constant C depends only on R and ω.

Proof. We first note that (3.40) still holds for the scattering problem described in the present lemma. Then one has

$$\int_{D_\delta\backslash\overline{D}_{\delta/2}}\sigma_l\mathbf{E}_\delta\cdot\overline{\mathbf{E}}_\delta\mathrm{d}\mathbf{x} \leqq C\|\nu\times\mathbf{E}_\delta^+\|_{\mathrm{TH}^{-1/2}_{\mathrm{div}}(\partial B_R)}\|\Lambda(\nu\times\mathbf{E}_\delta^+)\|_{\mathrm{TH}^{-1/2}_{\mathrm{div}}(\partial B_R)}$$
$$+\mathbb{R}_1+\mathbb{R}_2,$$

where by using (2.8) and (4.12) one further has

$$\begin{aligned}\mathbb{R}_1 &= \left|\Re\int_{\partial D_\delta}(\nu\times\overline{\mathbf{E}^i})\cdot(\nu\times\nu\times\mathbf{H}_\delta^+)\Big|_+ \mathrm{d}\sigma_x\right| = \left|\Re\int_{\partial D_\delta}\overline{\mathbf{E}^i}\cdot(\nu\times\mathbf{H}_\delta^+)\Big|_+ \mathrm{d}\sigma_x\right|\\ &\leqq \frac{1}{\omega}\left|\int_{S^0}\overline{\mathbf{E}^i}(\mathbf{z}_{\tilde{x}})\cdot\mathcal{L}_{S^0}^{\omega}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})\,\mathrm{d}\sigma_{\tilde{x}}\right| + \mathcal{O}(\delta\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}),\end{aligned}$$

and by using (4.11),

$$\begin{aligned}\mathbb{R}_2 &= \left|\Re\int_{\partial D_\delta}(\nu\times\overline{\mathbf{E}_\delta^+})\Big|_+\cdot(\nu\times\nu\times\mathbf{H}^i)\,\mathrm{d}\sigma_x\right| = \left|\Re\int_{\partial D_\delta}\overline{\mathbf{E}_\delta^+}\Big|_+\cdot(\nu\times\mathbf{H}^i)\,\mathrm{d}\sigma_x\right|\\ &\leqq \delta\left|\int_{S^0}\mathcal{M}_{S^0}^{\omega}[\tilde{\mathbf{a}}](\tilde{\mathbf{x}})\cdot\mathbf{H}^i(\mathbf{z}_{\tilde{x}})\,\mathrm{d}\sigma_{\tilde{x}}\right| + \mathcal{O}(\delta^2\|\tilde{\mathbf{a}}\|_{\mathrm{TH}^{-1/2}(\partial D)}).\end{aligned}$$

Combining the above estimates with (4.5), (4.6) and the definition of σ_l in (4.4), we can complete the proof. □

Proof of Theorem 4.1. First, by using (4.14) and (4.7) one obtains

$$\|\mathbf{A}_\infty^\delta(\hat{\mathbf{x}};\mathbf{p},\mathbf{d})\| \leqq C\Big(\varepsilon + \|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} + 1)\Big), \tag{4.30}$$

where $\tilde{\Phi}(\tilde{\mathbf{x}}) = \Phi(\mathbf{x}) := \nu\times\mathbf{E}_\delta\Big|_+(\mathbf{x})$ for $\mathbf{x}\in\partial D_\delta$. The following estimate can be obtained by using the result in Lemma 4.2 and a completely similar argument as that in the proof of Lemma 3.9,

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^j)} \leqq C\delta^{-(j+1)/2+\beta_j}\|\mathbf{E}_\delta\|_{L^2(D_\delta\backslash\overline{D_{\delta/2}})^3}, \quad j=0,1,2, \tag{4.31}$$

where $\beta_j = \min\{1, -j+r+s, -j+t+s\}$, $j=0,1,2$. On the other hand, by using (4.17), Lemma 4.3 and (4.30), one has

$$\begin{aligned}\int_{D_\delta^j\backslash\overline{D_{\delta/2}}}\|\mathbf{E}_\delta\|^2 d\mathbf{x} \leqq C\delta^{j-t}\Big(&\|\tilde{\Phi}\|^2_{\mathrm{TH}^{-1/2}(S^0)} + \delta^2\|\tilde{\Phi}\|^2_{\mathrm{TH}^{-1/2}(\partial D)}\\ &+ \|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}\\ &+\varepsilon+\delta\Big), \quad j=0,1,2.\end{aligned} \tag{4.32}$$

Inserting (4.32) back into (4.31), one can show

$$\begin{aligned}\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^j)} \leqq C\delta^{\beta_j-t/2-j/2-1/2}\Big(&\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \delta^{1/2}\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)}\\ &+ \|\tilde{\Phi}\|^{1/2}_{\mathrm{TH}^{-1/2}(S^0)} + \varepsilon^{1/2}+\delta^{1/2}\Big), \quad j=0,1,2.\end{aligned} \tag{4.33}$$

Noting that $\beta_2 \leqq \beta_1 \leqq \beta_0$ and $\beta_2 - t/2 \geqq 3/2$, one has for $\delta\in\mathbb{R}_+$ sufficiently small that

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} \leqq C\delta^{\beta_2-t/2-3/2}\Big(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \|\tilde{\Phi}\|^{1/2}_{\mathrm{TH}^{-1/2}(S^0)} + \varepsilon^{1/2}+\delta^{1/2}\Big). \tag{4.34}$$

Inserting (4.34) back into (4.33) (for $j = 0$), there holds

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} \leqq C\delta^{\beta_0-t/2-1/2}\Big(\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} + \|\tilde{\Phi}\|^{1/2}_{\mathrm{TH}^{-1/2}(S^0)} + \varepsilon^{1/2} + \delta^{1/2}\Big). \tag{4.35}$$

By $\beta_2 - t/2 \geqq 3/2$, it is directly verified that $\beta_0 - t/2 - 1/2 \geqq 1$ and hence

$$2(\beta_0 - t/2) - 1 \geqq \beta_0 - t/2. \tag{4.36}$$

Using (4.35) and (4.36), one readily infers for $\delta \in \mathbb{R}_+$ sufficiently small that

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(S^0)} \leqq C(\delta^{\beta_0-t/2} + \varepsilon). \tag{4.37}$$

Now by inserting (4.37) into (4.34), one has

$$\|\tilde{\Phi}\|_{\mathrm{TH}^{-1/2}(\partial D)} \leqq C\delta^{\beta_2-t/2-3/2}(\delta^{1/2} + \varepsilon^{1/2}) \leqq C(\delta^{\beta_2-t/2-1} + \delta^{2(\beta_2-t/2)-3} + \varepsilon). \tag{4.38}$$

Finally by plugging (4.37) and (4.38) into (4.30), we arrive at (4.8).

The proof is complete. □

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