Lecture 8. Proving the boundedness for optimal slice rank decompositions

Thomas Karam

Shanghai Jiao Tong University

16 July 2026

A systematic construction of different slice rank decompositions

- Assume that

$$\sum_{i=1}^{r} a_i(x)b_i(y,z) + \sum_{j=1}^{s} c_j(y)d_j(x,z) + \sum_{k=1}^{t} e_k(z)f_k(x,y)$$

  is a slice rank decomposition of some order-3 tensor T.
- Then for any fixed pair (i,j) and any function $e : [n_3] \to \mathbb{F}$, replacing b_i and d_j by

$$b_i' = b_i + c_j \otimes e \text{ and } d_j' = d_j - a_i \otimes e$$

  provides another decomposition of T.
- This produces a number of decompositions that is not bounded in terms of $(d, k, |\mathbb{F}|)$.

A boundedness result on slice rank decompositions

Theorem (K 2023)

Let $d \geq 3$, $k \geq 1$, let $\mathbb{F}$ be a finite field, and let T be an order-d tensor with slice rank k over $\mathbb{F}$. Then the slice rank decompositions

$$T(x_1, \dots, x_d) = \sum_{j=1}^{d} \sum_{i=1}^{r_j} a_{j,i}(x_j) b_{j,i}(\overline{x_j})$$

with minimal length $r_1 + \cdots + r_d = k$ obey the following two properties.

1. There are at most $d^k |\mathbb{F}|^{dk^2}$ possibilities for $((r_1, \dots, r_d), (a_{1,1}, \dots, a_{1,r_1}), \dots, (a_{d,1}, \dots, a_{d,r_d}))$.
2. For each of these possibilities, the functions $b_{j,i}$ between decompositions may only differ as in the previous construction (generalised to order-d tensors).

Describing the flexibility of $(d-1)$-variable functions

Definition

Let $d \geq 3$ be an integer. We say that an order-d decomposition is in *zero form* if the following are satisfied.

1. For every $j \in [d]$ and every $i \in [r_j]$ we can write
$$b_{j,i}(\overline{x_j}) = \sum_{J \subset [d] : j \in J} \sum_{s_{J \setminus \{j\}}} \left(\prod_{j' \in J \setminus \{j\}} a_{j', s_{j'}}(x_{j'}) \right) c_{J,j,i,s_{J \setminus \{j\}}}(x([d] \setminus J)).$$
2. For every $J \subset [d]$ with $2 \leq |J| \leq d$ and every $(s_{j'})_{j' \in J}$, we obtain a sum of 0 whenever we take the sum over all functions $c_{J,j,i,s_{J \setminus \{j\}}}$ where the sequence $s_{J \setminus \{j\}}$ completed to a sequence indexed by J by introducing the additional term $s_j = i$ is equal to $(s_{j'})_{j' \in J}$.

Describing the flexibility of $(d-1)$-variable functions

Proposition (K 2023)

Let $d \geq 3$ be an integer, and let $r_1, \dots, r_d$ be nonnegative integers. If a decomposition

$$\sum_{j=1}^{d} \sum_{i=1}^{r_j} a_{j,i}(x_j) b_{j,i}(\overline{x_j})$$

is equal to the zero tensor, then it is in zero form.

Describing the flexibility of $(d-1)$-variable functions

Proof:

- By making changes of bases on each of the d dimensions, we may assume that there exist integers $f(j,i)$ such that the functions $a_{j,i}$ are defined by $a_{j,i}(x_j) = 1$ if $x_j = f(j,i)$ and $a_{j,i}(x_j) = 0$ otherwise. We write
$$S_j = \{f(j,i) : i \in [r_j]\}$$
for each $j \in [d]$.
- The decomposition then becomes
$$\sum_{j=1}^{d} \sum_{i=1}^{r_j} 1_{x_j = f(j,i)} b_{j,i}(\overline{x_j}). \tag{1}$$
We then evaluate (1) at every $x \in [n_1] \times \cdots \times [n_d]$.

Describing the flexibility of $(d-1)$-variable functions

- If $x_1 \notin S_1, x_2 \notin S_2 \dots, x_d \notin S_d$, then there are no contributions to (1).
- If $x_1 \in S_1, x_2 \notin S_2, x_3 \notin S_3, \dots, x_d \notin S_d$, then the only contribution to (1) is from $1_{x_1 = f(1,i)} b_{1,i}(\overline{x_1})$ with $f(1,i) = x_1$, which must hence be 0.
- If $x_1 \in S_1, x_2 \in S_2, x_3 \notin S_3, x_4 \notin S_4, \dots, x_d \notin S_d$, then there are two contributions to (1), the sum of which must be 0: one from $1_{x_1 = f(1,i)} b_{1,i}(\overline{x_1})$ with $f(1,i) = x_1$, and one from $1_{x_2 = f(2,i)} b_{2,i}(\overline{x_2})$ with $f(2,i) = x_2$.

Describing the flexibility of $(d-1)$-variable functions

- More generally, if J is a subset of $[d]$, and we have that $x_j \in S_j$ if and only if $j \in J$, then the contributions to (1) (which again add to 0) are from the terms $1_{x_j = f(j,i)} b_{j,i}(\overline{x_j})$ with $f(j,i) = x_j$ and $j \in J$. This provides the desired description of the functions $b_{j,i}$ with $j \in [d]$ and $i \in [r_j]$.

The sunflower statement

Let T be an order-d tensor, and let A_j be a linear subspace of $\mathbb{F}^{n_j}$ for every $j \in [d]$. We write $T \in \mathcal{S}(A_1, \dots, A_d)$ if we can write

$$T(x_1, \dots, x_d) = \sum_{j=1}^{d} \sum_{i=1}^{r_j} a_{j,i}(x_j) b_{j,i}(\overline{x_j})$$

with $A_j = \langle a_{j,1}, \dots, a_{j,r_j} \rangle$ for every $j \in [d]$.

The sunflower statement

Definition

Let h be a positive integer, and let $S^1, \ldots, S^h$ be sets of integers. We say that $S^1, \ldots, S^h$ constitute a *sunflower* if there exist pairwise disjoint sets of integers $S^0, S^{1\prime}, \ldots, S^{h\prime}$ such that $S^\theta = S^0 \cup S^{\theta\prime}$ for every $\theta \in [h]$. The common intersection S^0 of $S^1, \ldots, S^h$ is called the center of the sunflower.

Definition

Let h be a positive integer, and let $A^1, \ldots, A^h$ be linear subspaces of some vector space V. We say that $A^1, \ldots, A^h$ constitute a *sunflower* if there exist linear subspaces $A^0, A^{1\prime}, \ldots, A^{h\prime}$ of V, all in direct sum, such that $A^\theta = A^0 \oplus A^{\theta\prime}$ for every $\theta \in [h]$.The common intersection A^0 of $A^1, \ldots, A^h$ is called the center of the sunflower.

The sunflower statement

Proposition (K 2023)

Let $d \geq 2$ be an integer, let $\mathbb{F}$ be a field, and let T be an order-d tensor over $\mathbb{F}$. Let $h > d$ be an integer. For every $\theta \in [h]$, let $A_1^\theta, \dots, A_d^\theta$ be linear subspaces such that $T \in \mathcal{S}(A_1^\theta, \dots, A_d^\theta)$. Suppose that for every $j \in [d]$ the subspaces A_j^θ with $\theta \in [d]$ constitute a sunflower, with center A_j^0. Then $T \in \mathcal{S}(A_1^0, \dots, A_d^0)$. In particular,

$$\operatorname{sr} T \leq \dim A_1^0 + \dots + \dim A_d^0.$$

The sunflower statement

- Without loss of generality we may assume that there exist integers $f(j,i)$ such that the functions $a_{j,i}$ are defined by $a_{j,i}(x_j) = 1$ if $x_j = f(j,i)$ and $a_{j,i}(x_j) = 0$ otherwise.
- The assumption then states that for every $j \in [d]$ the sets

$$S_j^\theta = \{f(j,i) : 1 \leq i \leq r_j^\theta\}$$

with $\theta \in [h]$ constitute a sunflower.

The sunflower statement

- Let $x \in [n_1] \times \cdots \times [n_d]$ such that $T(x_1, \ldots, x_d) \neq 0$. Then for each $\theta \in [h]$ we must necessarily have $x_j \in S_j^{\theta}$ for some $j \in [d]$.
- Because $h > d$, by the pigeonhole principle there exists some $j \in [d]$ and two distinct values θ_1, θ_2 of θ such that $x_j \in S_j^{\theta_1}$ and $x_j \in S_j^{\theta_2}$.
- Since $S_j^{\theta_1} \cap S_j^{\theta_2} = S_j^0$, this ensures $x_j \in S_j^0$.
- This establishes that we may write T as desired.

The sunflower statement

Corollary (K 2023)

Let $d \geq 2$ be an integer, let $\mathbb{F}$ be a field, and let T be an order-d tensor over $\mathbb{F}$. Let $h \geq 1$ be an integer, and consider for every $\theta \in [h]$ some *minimal-length* decomposition of T. For every $j \in [d]$ let W_j be a linear subspace of $\mathbb{F}^{n_j}$. Suppose that the linear subspaces $A_j^\theta = \langle a_{j,1}, \dots, a_{j,r_j^\theta} \rangle$ constitute a sunflower with center W_j for every $j \in [d]$. If

$$\dim W_1 + \dots + \dim W_d < \operatorname{sr} T$$

then $h \leq d$.

Proof sketch of the boundedness for one-variable functions

Let T be an order-d tensor with slice rank k over $\mathbb{F}$. Consider a set of minimal-length slice rank decompositions indexed by $\theta \in [h]$. We construct a tree where all vertices are subsets of $[h]$ and which satisfies the following properties.

1. The root is the set $[h]$ itself.
2. The tree has depth k.
3. Every non-leaf vertex is the union of its immediate descendants.
4. Every vertex has at most $d|\mathbb{F}|^{dk}$ immediate descendants.
5. Every leaf is a set with size at most 1.

Obtaining such a tree indeed suffices to conclude, as these properties together imply that the integer h is at most $(d|\mathbb{F}|^{dk})^k = d^k|\mathbb{F}|^{dk^2}$, and hence that the number of d-tuples of linear subspaces $(A_1^\theta, \ldots, A_d^\theta)$ is at most $d^k|\mathbb{F}|^{dk^2}$.

Conclusion

Thank you for listening !

thomaskaram@sjtu.edu.cn