Lecture 7. Proving the extension of the submatrix property to tensors

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A high-rank tensor has a small high-rank subtensor

Theorem (K 2022)

Let $d \geq 2$ be an integer, and let R be a non-empty family of partitions of $[d]$. There exist functions $F_{d,R} : \mathbb{N} \to \mathbb{N}$ and $G_{d,R} : \mathbb{N} \to \mathbb{N}$ such that if T is an order d tensor such that $R\text{rk}\, T \geq G_{d,R}(l)$ then there exist $X_1 \subset [n_1], \dots, X_d \subset [n_d]$ each with size at most $F_{d,R}(l)$ such that

$$R\text{rk}\, T(X_1 \times \cdots \times X_d) \geq l.$$

The proof in the matrix case

- We characterise $\operatorname{rk} A$ as the smallest $k \geq 0$ such that there exist vectors $R_1, \ldots, R_k \in \mathbb{F}^{n_2}$ such that $A_x \in \langle R_1, \ldots, R_k \rangle$ for each $x \in [n_1]$.
- We extract a basis $(A_x : x \in X)$ of the linear subspace $\langle A_x : x \in [n_1] \rangle$ spanned by all rows of A, with a set X of size $\dim\langle A_x : x \in [n_1] \rangle = \operatorname{rk} A$.
- It follows from the two previous steps that $\operatorname{rk} A(X \times [n_2]) = \operatorname{rk} A$. Iterating again on the second coordinate we are done.

The proof in the order-3 tensor rank case 1/3

We characterise the tensor rank of an order 3 tensor T as being the smallest number of rank 1 matrices which suffice to span all its slices T_x.

Definition: Spanning rank

Let V be a vector space over $\mathbb{F}$, and let $S, F \subset V$. We say that the *spanning rank* of F with respect to S is the smallest nonnegative integer k such that there exist vectors $v_1, \dots, v_k \in S$ satisfying $F \subset \langle v_1, \dots, v_k \rangle$. For $F \subset V$, we will denote by $\mathrm{sprk}_S\, F$ the spanning rank of F.

Let S_2 be the set of matrices $[n_2] \times [n_3] \to \mathbb{F}$ with rank 1. For $T : [n_1] \times [n_2] \times [n_3]$ an order d tensor, we can write

$$\mathrm{tr}\, T = \mathrm{sprk}_{S_2} \{T_x : x \in [n_1]\}.$$

The proof in the order-3 tensor rank case 2/3

Lemma (K 2022)

Let $k \geq 1$ be an integer, let V be a vector space over $\mathbb{F}$, and let S and F be families of elements of V. Assume that $\operatorname{sprk}_S F \geq k$. Then there exists a subfamily F' of F with size at most k such that $\operatorname{sprk}_S F' \geq k$.

Proof: We distinguish two cases.

- Case 1: The linear subspace $\langle F \rangle$ satisfies $\dim\langle F \rangle \geq k$. Then we take F' to be a linearly independent family of k elements of F. Because $\dim\langle F' \rangle = k$, in particular $\operatorname{sprk}_S F' \geq k$.
- Case 2: We have $\dim\langle F \rangle \leq k$. Then we take F' to be a maximal linearly independent family of elements of F. The family F' has size at most k, and since $\langle F \rangle = \langle F' \rangle$, a family of elements of S spans all elements of F iff it spans all elements of F', so $\operatorname{sprk}_S F' = \operatorname{sprk}_S F$.

The proof in the order-3 tensor rank case 3/3

By the previous two steps we obtain a set X with size at most k such that

$$\mathrm{sprk}_{S_2}\{T_x : x \in X\} = k$$

so $\mathrm{tr}\, T(X \times [n_2] \times [n_3]) = k$. Iterating twice more we find Y, Z with size at most k and such that

$$\mathrm{tr}\, T(X \times Y \times Z) = k. \quad \square$$

The proof for order-d tensors is similar for any $d \geq 3$.

Lower bounds on the slice rank using separated slices

If a tensor has a large well-separated set of slices then it has high slice rank.

Proposition (K 2022)

Let $T : [n_1] \times [n_2] \times [n_3] \to \mathbb{F}$ be an order-3 tensor, and let $l \geq 1$ be an integer. If there exist $x_1, \dots, x_l \in [n_1]$ such that

$$\mathrm{rk}\left(\sum_{i=1}^{l} a_i T_{x_i}\right) \geq l$$

for every $(a_1, \dots, a_l) \in \mathbb{F}^l \setminus \{0\}$, then $\mathrm{sr}\, T \geq l$.

Lower bounds on the slice rank using separated slices

Basic proof idea. The general form of a slice rank decomposition is

$$T(x,y,z) = \sum_{i=1}^{r} a_i(x) b_i(y,z) + \sum_{j=1}^{s} c_j(y) d_j(x,z) + \sum_{k=1}^{t} e_k(z) f_k(x,y).$$

When fixing the first coordinate x, this becomes

$$T_x(y,z) = \sum_{i=1}^{r} a_{i,x} b_i(y,z) + \sum_{j=1}^{s} c_j(y) d_{j,x}(z) + \sum_{k=1}^{t} e_k(z) f_{k,x}(y).$$

All terms from the last two parts have rank at most 1 and can be viewed as "error terms".

Lower bound on the slice rank when slices have small rank

If a tensor has all its slices (in all three directions) with bounded rank, then its tensor rank and slice rank are "equivalent".

Proposition (K 2022)

Let $T : [n_1] \times [n_2] \times [n_3] \to \mathbb{F}$ be an order 3 tensor. Let $m \geq 1$ be an integer. Assume that for all $x \in [n_1], y \in [n_2], z \in [n_3]$ we have $\operatorname{rk} T_x, \operatorname{rk} T_y, \operatorname{rk} T_z \leq m$. Then $\operatorname{tr} T \leq m(\operatorname{sr} T)^2$.

Lower bound on the slice rank when slices have small rank

Proof:

- We start from

$$T(x,y,z) = \sum_{i=1}^{r} a_i(x) b_i(y,z) + \sum_{j=1}^{s} c_j(y) d_j(x,z) + \sum_{k=1}^{t} e_k(z) f_k(x,y)$$

  with $r + s + t = \mathrm{sr}\, T$.

- By Gaussian elimination there exist a subset $X_1 \subset [n_1]$ with size r and functions $a_i^* : [n_1] \to \mathbb{F}$ supported inside X_1 with

$$a_{i'}^*.a_i := \sum_x a_{i'}^*(x) a_i(x) = \delta_{i,i'}.$$

Lower bound on the slice rank when slices have small rank

- Applying a_i^* to the decomposition we get

$$(a_i^*.T)(y,z) = b_i(y,z) + \sum_{j=1}^{s} c_j(y)(a_i^*.d_j)(z) + \sum_{k=1}^{t} e_k(z)(a_i^*.f_k)(y).$$

- The LHS has rank at most rm and the two last terms of the RHS have rank at most $s+t$.

Application

A tensor with high slice rank has a small subtensor with high slice rank.

Proposition (K 2022)

Let $T : [n_1] \times [n_2] \times [n_3] \to \mathbb{F}$ be an order-3 tensor, and let $l \geq 1$ be a positive integer. If $\operatorname{sr} T \geq 51 l^3$ then there exist $X \subset [n_1]$, $Y \subset [n_2]$, $Z \subset [n_3]$ with size at most $48 l^3$ such that $\operatorname{sr} T(X \times Y \times Z) \geq l$.

Ingredients of the proof.

- Both of the previous lower bounds on the slice rank.
- A decomposition $T = S + U$ that reduces to these two cases.
- The tensor rank case: a tensor with tensor rank k has a $k \times k \times k$ subtensor with tensor rank k.

Conclusion

Thank you for listening !