Lecture 6. Formulations of basic properties of the ranks of tensors

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15 July 2026

A high-rank tensor has a small high-rank subtensor

- A matrix with rank k must contain a $k \times k$ submatrix with rank k.
- Extension to the tensor rank: a tensor with tensor rank k has a $k \times k \times \cdots \times k$ subtensor with tensor rank k.
- This property fails already for the slice rank of order-3 tensors as shown by a counterexample with slice rank 4 constructed by Gowers.

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The following generalisation can nonetheless be established.

Theorem (K 2022)

Let $d \geq 2$ be an integer, and let R be a non-empty family of partitions of $[d]$. There exist functions $F_{d,R} : \mathbb{N} \to \mathbb{N}$ and $G_{d,R} : \mathbb{N} \to \mathbb{N}$ such that if T is an order d tensor such that $R\text{rk } T \geq G_{d,R}(l)$ then there exist $X_1 \subset [n_1], \ldots, X_d \subset [n_d]$ each with size at most $F_{d,R}(l)$ such that

$$R\text{rk } T(X_1 \times \cdots \times X_d) \geq l.$$

The number of rank decompositions of a matrix

- A rank-k matrix may only have one rank decomposition with length k, up to a change of basis.
- If $\mathbb{F}$ is a finite field, then the number of decompositions with length k of a rank-k matrix is equal to the number

$$(|\mathbb{F}|^k - 1)(|\mathbb{F}|^k - |\mathbb{F}|) \dots (|\mathbb{F}|^k - |\mathbb{F}|^{k-1}) = |\mathbb{F}|^{k^2} \prod_{i=1}^{k} (1 - |\mathbb{F}|^{-i})$$

  of bases of $\mathbb{F}^k$, which is between $c|\mathbb{F}|^{k^2}$ and $|\mathbb{F}|^{k^2}$ for some absolute constant $c > 0$.
- How does this fact generalise to the slice rank?

Constructions of different slice rank decompositions 1/5

- Consider a slice rank decomposition

$$T(x,y,z) = \sum_{i=1}^{r} a_i(x) b_i(y,z) + \sum_{j=1}^{s} c_j(y) d_j(x,z) + \sum_{k=1}^{t} e_k(z) f_k(x,y)$$

  with minimal length $r+s+t$.
- We can view it as

$$T(x,y,z) = M_1(x,(y,z)) + M_2(y,(x,z)) + M_3(z,(x,y))$$

  for some matrices M_1, M_2, M_3 with ranks r, s, t.
- It then suffices to rewrite the decomposition of any of the matrices M_1, M_2, M_3 using a change of basis.

Constructions of different slice rank decompositions 2/5

- The previous construction only involved rewriting any of the individual three parts. However, the parts are not determined.
- Suppose for instance that the tensor rank of T is equal to its slice rank, and

$$T(x,y,z) = \sum_{i=1}^{k} a_i(x)c_i(y)e_i(z)$$

is a tensor rank decomposition of T with minimal length k.
- Then for any tripartition $\{I, J, K\}$ of $[k]$ the decomposition

$$\sum_{i \in I} a_i(x)(c_i e_i(y,z)) + \sum_{j \in J} c_j(y)(a_j e_j(x,z)) + \sum_{k \in K} e_k(z)(a_k c_k(x,y))$$

is a slice rank decomposition of T with minimal length.

Constructions of different slice rank decompositions 3/5

- A tensor $T : [k]^3 \to \mathbb{F}$ can be written in at least 3 different ways by writing it as a sum of its slices of each type, i.e. by decomposing it as

$$T(x, y, z) = \sum_{i=1}^{k} 1_{x=i} T(i, y, z)$$

and similarly after exchanging the roles of the three coordinates.

- If T has slice rank k, then these 3 decompositions are each minimal-length decompositions.
- Such tensors exist. Indeed, if the field $\mathbb{F}$ is finite, then the number of tensors $[k]^3 \to \mathbb{F}$ with slice rank at most $k-1$ is at most

$$k^3 |\mathbb{F}|^{(k-1)(k^2+k)} = (k^3/|\mathbb{F}|^k)|\mathbb{F}|^{k^3}.$$

The ratio tends to 0 as k tends to infinity, which shows that random tensors have slice rank k.

Constructions of different slice rank decompositions 4/5

- Assume that

$$\sum_{i=1}^{r} a_i(x)b_i(y,z) + \sum_{j=1}^{s} c_j(y)d_j(x,z) + \sum_{k=1}^{t} e_k(z)f_k(x,y)$$

  is a slice rank decomposition of T.

- Then for any fixed pair (i,j) and any function $e : [n_3] \to \mathbb{F}$, replacing b_i and d_j by

$$b_i' = b_i + c_j \otimes e \text{ and } d_j' = d_j - a_i \otimes e$$

  provides another decomposition of T.

- This produces a number of decompositions that is not bounded in terms of $(d, k, |\mathbb{F}|)$.

Constructions of different slice rank decompositions 5/5

- We can take this idea further. Assume that

$$\sum_{i=1}^{r} a_i(x)b_i(y,z) + \sum_{j=1}^{s} c_j(y)d_j(x,z) + \sum_{k=1}^{t} e_k(z)f_k(x,y)$$

  is a slice rank decomposition of T.
- Then for any fixed triple (i,j,k) and any coefficients $\lambda_1, \lambda_2, \lambda_3$ with sum 0, replacing b_i,d_j,f_k by

$$b_i' = b_i + \lambda_1 c_j \otimes e_k,\ d_j' = d_j + \lambda_2 a_i \otimes e_k,\ f_k' = f_k + \lambda_3 a_i \otimes c_j$$

  provides another decomposition of T.
- However, doing so ultimately reduces to doing three of the previous operations, so this is not a genuinely new way in which more decompositions can arise.

A boundedness result on slice rank decompositions

Theorem (K 2023)

Let $d \geq 3$, $k \geq 1$, let $\mathbb{F}$ be a finite field, and let T be an order-d tensor with slice rank k over $\mathbb{F}$. Then the slice rank decompositions

$$T(x_1, \dots, x_d) = \sum_{j=1}^{d} \sum_{i=1}^{r_j} a_{j,i}(x_j) b_{j,i}(\overline{x_j})$$

with minimal length $r_1 + \cdots + r_d = k$ obey the following two properties.

1. There are at most $d^k |\mathbb{F}|^{dk^2}$ possibilities for $((r_1, \dots, r_d), (a_{1,1}, \dots, a_{1,r_1}), \dots, (a_{d,1}, \dots, a_{d,r_d}))$.
2. For each of these possibilities, the functions $b_{j,i}$ between decompositions may only differ as in the previous construction (generalised to order-d tensors).

Broader comments

- The rank on matrices generalises to several notions of rank on tensors.
- The extension to tensors of a basic matrix property is often false in its most obvious form.
- Nonetheless, a relaxation of that property holds which is still substantial, not too difficult to state, and in the spirit of the original property.
- There is another kind of rectification (not discussed today): keep the original full property, at the cost of it only applying to a wide concisely described class of tensors rather than to all tensors.

Conclusion

Thank you for listening !