Lecture 5: The linear group of transformations on tensors, and the subrank

Thomas Karam

Shanghai Jiao Tong University

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Canonical forms

Assume that the field $\mathbb{F}$ is finite.

- The group $\mathrm{GL}_n(\mathbb{F})$ has size between $c|\mathbb{F}|^{n^2}$ and $|\mathbb{F}|^{n^2}$ for some absolute constant $c > 0$.
- The number of tensors $[n]^d \to \mathbb{F}$ is $|\mathbb{F}|^{n^d}$, which is not too much larger when $d = 2$, but becomes much larger when $d > 3$.
- A rank-k matrix is always equivalent to a J_k block matrix. However, there is no analogous result for the ranks of tensors when $d > 3$.

Zariski-closed sets

In the following, we say that a notion of rank on order-d tensors is Zariski-closed if for every integer $k \geq 0$ the set of tensors $[n]^d \to \mathbb{F}$ with rank at most k is Zariski-closed.

- The matrix rank is Zariski-closed. Indeed the set is the set of matrices for which every $(k+1) \times (k+1)$ submatrix has determinant 0.
- The slice rank is always Zariski-closed for all $d \geq 3$ (Sawin Tao 2016). The proof uses the dual formulation of the slice rank.
- However, the partition rank is believed to not be Zariski-closed for any $d \geq 4$. The analogous notion for polynomials (the strength) is not Zariski-closed (Ballico Bik Oneto Ventura 2022).

The restriction relation

Definition

Let $d \geq 2$ be an integer, let $T : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ and let $S : [m_1] \times \cdots \times [m_d] \to \mathbb{F}$ be order-d tensors. We say that S *is a restriction of* T if after making changes of basis on the rows, columns, and depth of T we can obtain S as a subtensor.

We write $S \leq T$.

The restriction relation

The restriction relation is a preorder.

- It is reflexive: we have $T \leq T$.
- It is transitive: if $S \leq T$ and $R \leq S$ then $R \leq T$.
- It is not antisymmetric: for instance if A, B are two invertible $n \times n$ matrices then $A \leq B$ and $B \leq A$.

The subrank of tensors

Definition

Let $d \geq 2$ be an integer and let $T : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$ be an order-d tensor. We say that the *subrank* of T is the largest nonnegative integer k such that the identity tensor I_k satisfies $I_k \leq T$.

We denote by $Q(T)$ the subrank of T.

The subrank of tensors

- In a sense the subrank is the opposite of the tensor rank. The tensor rank of T is the smallest integer k such that $T \leq I_k$.
- For $d = 2$, the subrank is equal to matrix rank, due to the canonical form of a rank-k matrix.

Subrank versus R-rank

Proposition

Let R be a non-empty family of partitions of $[d]$. If R does not contain the trivial partition $\{[d]\}$, then the subrank of T is at most the R-rank of T for every order-d tensor T.

The subrank of tensors

Proof:

- If $S \leq T$ then $\mathrm{Rrk}\, S \leq \mathrm{Rrk}\, T$.
- Since $\{[d]\} \notin R$, for each k the R-rank of the identity tensor I_k is at least the partition rank of I_k, which is equal to k by a result of Naslund (2017).
- We conclude by taking $k = Q(T)$ and $S = I_k$.

Comparing the subrank with other notions

- The subrank is always at most the partition rank, so is at most n. It can hence be much smaller than the tensor rank.
- The generic subrank of an $n \times n \times n$ tensor is $\Theta(\sqrt{n})$. (Proof: counting argument.) Meanwhile, the generic partition rank is n.
- So the subrank is generically much smaller than all notions of R-rank.

Conclusion

Thank you for listening !