Lecture 4: A connection between the slice covering number and the slice rank

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The Sawin-Tao connection

If $d \geq 2$ is an integer, and A is an order-d lattice subset, then we say that A is an *antichain* if no two distinct elements $x, y \in A$ satisfy $x_j \leq y_j$ for every $j \in [d]$.

Proposition (Sawin-Tao 2016)

Let $d \geq 2$ be an integer, and let T be an order-d tensor. If the support of T is contained in an antichain, then the slice rank of T is equal to the smallest number of slices (i.e. sets of the type $\{x_j = a\}$) that suffice to cover the support of T.

Outline of the proof of the Sawin-Tao connection

Let T be an order-d tensor, let Γ be the support of T, and let Γ' be the set of maximal elements of Γ for the product ordering.
If A is a set of points, let $\operatorname{sc} A$ for the smallest number of slices that suffice to cover A.

- Establish that $\operatorname{sr} T \leq \operatorname{sc} \Gamma$.
- Establish that $\operatorname{sr} T \geq \operatorname{sc} \Gamma'$.
- If the support Γ is an antichain, then $\Gamma = \Gamma'$, and therefore $\operatorname{sr} T = \operatorname{sc} \Gamma$.

Upper bound

- Suppose
$$\Gamma \subset \bigcup_{1 \leq j \leq d} \bigcup_{1 \leq i \leq r_j} \{x_j = a_{j,i}\}$$
for some $r_1, \ldots, r_d$, some $a_{j,i}$ with $j \in [d]$ and $i \in [r_j]$.
- For every $(x_1, \ldots, x_d) \in \Gamma$, let $N(x_1, \ldots, x_d) \geq 1$ be the number of slices on the right-hand side that contain $(x_1, \ldots, x_d)$. We define a tensor S as follows: if $(x_1, \ldots, x_d) \notin \Gamma$ then $S(x_1, \ldots, x_d) = 0$, and if $x \in \Gamma$ then
$$S(x_1, \ldots, x_d) = T(x_1, \ldots, x_d)/N(x_1, \ldots, x_d).$$
- To conclude $\operatorname{sr} T \leq \operatorname{sc} \Gamma$, it suffices to write
$$T(x_1, \ldots, x_d) = \sum_{j=1}^{d} \sum_{i=1}^{r_j} 1_{x_j = a_{j,i}} S(x_1, \ldots, x_{j-1}, a_{j,i}, x_{j+1}, \ldots, x_d).$$

Outline of the proof of the lower bound

- Dual formulation of the slice rank.
- Row-echelon form.
- Disjointness of Γ' and the product of pivots.
- Conclusion.

Dual formulation

Suppose that T is an order-d tensor with slice rank k. Then there exist subspaces $U_1 \subset \mathbb{F}^{n_1}, \ldots, U_d \subset \mathbb{F}^{n_d}$ satisfying

$$(u_1 \otimes \cdots \otimes u_d).T = 0$$

for all $u_1 \in U_1, \ldots, u_d \in U_d$ and such that

$$\dim U_1 + \cdots + \dim U_d = n_1 + \cdots + n_d - k.$$

We write $\rho_j = \dim U_j$ for every $j \in [d]$.

Dual formulation

Proof: Write a minimal-length slice rank decomposition

$$T(x_1, \ldots, x_d) = \sum_{j=1}^{d} \sum_{i=1}^{r_j} a_{j,i}(x_j) b_{j,i}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d),$$

and choose

$$U_j = \langle a_{j,1}, \ldots, a_{j,r_j} \rangle^{\perp}$$

for every $j \in [d]$.

Row-echelon form

- For every $j \in [d]$ we find a basis $(u_{j,1}, \dots, u_{j,\rho_j})$ of U_j that is in row-echelon form in the standard basis $(e^*_{j,1}, \dots, e^*_{j,n_j})$ of $(\mathbb{F}^{n_j})^*$.
- We write $\pi_{j,1} \leq \dots \leq \pi_{j,\rho_j}$ for the pivot coordinates in this basis, that is, we have $u_{j,i} \in \langle e^*_{j,\pi_{j,i}}, \dots, e^*_{j,\pi_{j,\rho_j}} \rangle$ for every $j \in [d]$, $i \in [\rho_j]$, and the coefficient of $e^*_{j,\pi_{j,i}}$ is equal to 1.
- We write $\Pi_j = \{\pi_{j,1}, \dots, \pi_{j,\rho_j}\}$ for every $j \in [d]$.

Disjointness of Γ' and the product of pivots

Key observation: the set Γ' is disjoint from $\Pi_1 \times \cdots \times \Pi_d$.
Proof: Assume for contradiction that this is not the case.

- Then there exists $(\pi_{1,i_1}, \ldots, \pi_{d,i_d}) \in \Gamma'$.
- By maximality of Γ', $(\pi_{1,i_1}, \ldots, \pi_{d,i_d})$ is the only element in the intersection of Γ with $\prod_{j=1}^{d} \{\pi_{j,i_j}, \ldots, n_j\}$.
- The product $u_{1,i_1} \otimes \cdots \otimes u_{d,i_d}$ is hence equal to the sum of the product $e^*_{1,\pi_{1,i_1}} \otimes \cdots \otimes e^*_{d,\pi_{d,i_d}}$ and of products $e^*_{1,v'_1} \otimes \cdots \otimes e^*_{d,v'_d}$ with $(v'_1, \ldots, v'_d) \notin \Gamma$.
- Therefore, we obtain

$$\langle u_{1,i_1} \otimes \cdots \otimes u_{d,i_d}, T \rangle = T(\pi_{1,i_1}, \ldots, \pi_{d,i_d}) \neq 0,$$

a contradiction.

Conclusion

- For each $j \in [d]$ we define Γ_j to be the set of elements of Γ with jth coordinate outside of Π_j.
- By the disjointness observation, $\Gamma' \subset \bigcup_{1 \leq j \leq d} \Gamma_j$.
- By definition of Γ_j we have $\operatorname{sc} \Gamma_j = n_j - \rho_j$ for each $j \in [d]$.
- The desired bound follows.

Conclusion

Thank you for listening !