Lecture 3: Lattice coverings: basic questions, results and proofs

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The Sawin-Tao connection

If $d \geq 1$ is an integer, and A is an order-d lattice subset, then we say that A is an *antichain* if no two distinct elements $x, y \in A$ satisfy $x_j \leq y_j$ for every $j \in [d]$.

Proposition (Sawin-Tao 2016)

Let $d \geq 2$ be an integer, and let T be an order-d tensor. If the support of T is contained in an antichain, then the slice rank of T is equal to the smallest number of slices (i.e. sets of the type $\{x_j = a\}$) that suffice to cover the support of T.

Covering number at most 1

Let $d \geq 1$ be an integer.

- An *order-d lattice subset* is defined to be a subset of $[n_1] \times \cdots \times [n_d]$.
- If B is a subset of $[d]$, then we say that a *B-subspace* is a maximal subset S of $[n_1] \times \cdots \times [n_d]$ such that $x_i = y_i$ whenever $x, y \in S$ and $i \in B^c$, and say that it has *order* $|B|$.
- If M is a non-empty subset of the power set $\mathcal{P}([d])$ of $[d]$, then we define an *M-subspace* to be a B-subspace for some $B \in M$.
- If A is furthermore a subset of $[n_1] \times \cdots \times [n_d]$ then we say that A has *M-covering number at most* 1 if A is contained in an M-subspace.

Covering number at most k

- The *M-covering number* of A, denoted by $\mathrm{Mc}(A)$, is the smallest nonnegative integer k such that A is contained in a union

$$A_1 \cup \cdots \cup A_k$$

  of k subsets $A_1, \dots, A_k$ of $[n_1] \times \cdots \times [n_d]$ which each have M-covering number at most 1.
- For instance, for $d = 2$, a $\{1\}$-subspace is a horizontal line, a $\{2\}$-subspace is a vertical line, and any set contained in $\{x = 1\} \cup \{y = 1\}$ has $\{\{1\}, \{2\}\}$-covering number at most 2.

Diagonal additivity for covering numbers

Proposition

Let $d \geq 1$ be an integer, and let $M \subset \mathcal{P}([d])$ be non-empty. If A_1 and A_2 are order-d lattice subsets in diagonal sum and M does not contain $[d]$, then

$$\mathrm{Mc}(A_1 \cup A_2) = \mathrm{Mc}(A_1) + \mathrm{Mc}(A_2).$$

Proof: A B-subspace with $B \neq [d]$ can intersect at most one of the blocks.

Small sublattices with high covering number

Proposition (K 2023)

Let $d, l \geq 1$ be integers, and let $M \subset \mathcal{P}([d])$ be non-empty. If A is an order-d lattice subset such that $\mathrm{Mc}(A) \geq |M|l$, then there exist $X_1 \subset [n_1], \ldots, X_d \subset [n_d]$ each with size at most l such that

$$\mathrm{Mc}(A(X_1 \times \cdots \times X_d)) \geq l.$$

The key ingredient: independent sets

Let $d \geq 1$ be an integer, let $M \subset \mathcal{P}([d])$ be non-empty, and let A be an order-d lattice subset.

- We say that A is *M-independent* if whenever x, y are two distinct elements of A there is no M-subspace which contains both x and y.
- We note that if A is M-independent then every subset of A is again M-independent, and that $\mathrm{Mc}(A) = |A|$.
- For a general order-d lattice subset A, we define the *M-independence number* $I_M(A)$ of A to be the size of the largest M-independent subset of A.

Key lemma: A set with high covering number contains a large independent set

Lemma

Let $d \geq 1$ be an integer, let $M \subset \mathcal{P}([d])$ be non-empty, and let A be an order-d lattice subset. Then $I_M(A) \geq \mathrm{Mc}(A)/|M|$.

Proof: Construct an independent set by taking a point x_1 of A, then discarding all points of A in some M-subspace that contains x_1, then repeating until there are no points left.

Boundedness of the number of covering decompositions

Theorem (K 2023)

Let $d, l \geq 1$ be integers, and let $M \subset \mathcal{P}([d])$ be non-empty. If A is an order-d lattice subset such that $\mathrm{Mc}(A) = l$ then there are at most $(|M| l^d)^l$ l-tuples $(S_1, \dots, S_l)$ such that

$$A \subset S_1 \cup \dots \cup S_l$$

and the set S_i is an M-subspace for every $i \in [l]$.

The lower bound from the diagonal example

The bound previously obtained is not too far from the optimal bound.

Example

Let $d, l \geq 1$ be integers. Assume that $\min(n_1, \ldots, n_d) \geq l$. We take $A = \{(i, \ldots, i) : i \in [l]\}$. Let M be a non-empty subset of $\mathcal{P}([d])$ which does not contain $[d]$. Then the number of possible l-tuples $(S_1, \ldots, S_l)$ is equal to $l!|M|^l$.

Proof: No two points of A can be contained in the same M-subspace, so the M-covering number of A is equal to l. For each point $x \in A$, there are $|M|$ possible choices for an M-subspace containing x.

Bounded-size sublattices with full covering number

Theorem (K 2023)

Let $d, l \geq 1$ be integers, and let $M \subset \mathcal{P}([d])$ be non-empty. If A is an order-d lattice subset such that $\mathrm{Mc}(A) \geq l$ then there exist $X_1 \subset [n_1], \ldots, X_d \subset [n_d]$ each with size at most $O((\max(|M|, d) + 1)^{d(l+d)})$ such that

$$\mathrm{Mc}(A(X_1 \times \cdots \times X_d)) \geq l.$$

Main ideas in the proof

Consider the special case where the M-subspaces are the slices and $d = 3$.

- Extreme case: A slice is forced. For instance, $l = 2$ and A contains the points
$$(1,1,1), \ldots, (10,10,1):$$
the slice $\{z = 1\}$ must appear in the decomposition.
- Less extreme case: A line is forced. For instance, $l = 2$ and A contains the points
$$(1,1,1), \ldots, (10,1,1):$$
at least one of the slices $\{y = 1\}$ or $\{z = 1\}$ must appear in the decomposition.
- If neither of the above holds, then $|A| \leq l^3$. But then there are at most 3^{l^3} ways of covering the points of A (without wasting a slice).

Conclusion

Thank you for listening !