Lecture 2: The slice rank of tensors and a solution to the cap-set problem

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13 July 2026

Birth of the slice rank

Problem

Let $A \subset \mathbb{F}_3^n$ such that the only $(x, y, z) \in A \times A \times A$ satisfying $x + y + z = 0$ are the triples (x, x, x). How large can A be?

- Trivial bound $|A| \leq 3^n$.
- Meshulam (1995): $|A| \leq O(3^n/n)$ using Fourier density-increment methods.
- Bateman and Katz (2012): $|A| \leq O(3^n/n^{1+a})$ for some $a > 0$.
- Croot-Lev-Pach and Ellenberg-Gijswijt (2016), both published in the Annals of Mathematics: $|A| \leq O(C^n)$ for some $C < 3$ (about 2.756).
- Proof shortly after reformulated in a more symmetric way by Tao in terms of the slice rank.

The cap-set breakthrough

Theorem (Croot-Lev-Pach, Ellenberg-Gijswijt 2016)

There exists $C < 3$ such that the following holds. If $A \subset \mathbb{F}_3^n$ is such that $A \times A \times A$ contains no solution to $x + y + z = 0$ except solutions of the type (x, x, x) then $|A| = O(C^n)$.

Proof strategy for the cap-set problem

- If $A \subset \mathbb{F}_3^n$ satisfies the assumption, then

$$1_{x+y+z=0^n} = \sum_{a \in A} 1_{x=a} 1_{y=a} 1_{z=a}$$

  on $A \times A \times A$.
- Part 1: The right-hand side has slice rank $|A|$.
- Part 2: The left-hand side has slice rank $O(C^n)$ with $C < 3$.
- Therefore, $|A| = O(C^n)$.

The slice rank of a diagonal tensor

Proposition

Let $d \geq 2$, and let $T : A^d \to \mathbb{F}$ be an order-d tensor such that $T(x_1, \ldots, x_d) \neq 0$ if and only if $x_1 = \cdots = x_d$. Then the slice rank of T is equal to $|A|$.

For the application to the cap-set problem, the $d = 3$ case suffices. However, the proof is similar for all $d \geq 3$.

The slice rank of a diagonal tensor

Proof:

- Let us prove the case $d = 3$. The case $d \geq 3$ is similar by induction on d.
- The slice rank of T is at most $|A|$, since

$$T(x, y, z) = \sum_{a \in A} 1_{x=a} 1_{y=z=a}.$$

  So it suffices to prove that the slice rank of T is at least $|A|$.
- Let k be the slice rank of T. There exists a decomposition of the type

$$T(x, y, z) = \sum_{i=1}^{r} a_i(x) b_i(y, z) + \sum_{j=1}^{s} c_j(y) d_j(x, z) + \sum_{k=1}^{t} e_k(z) f_k(x, y),$$

  where $r + s + t = k$.

The slice rank of a diagonal tensor

- For each $i \in A$ let $\delta_i : A \to \mathbb{F}$ be defined by $\delta_i(u) = 1$ if $u = i$ and $\delta_i(u) = 0$ otherwise.
- There exists a subset $I \subset A$ with size $|A| - r$ such that

$$\{a_i : i \in [r]\} \cup \{\delta_i : i \in I\}$$

constitutes a basis of the linear space of functions $A \to \mathbb{F}$.

The slice rank of a diagonal tensor

- If $f, g : A \to \mathbb{F}$, let $f.g \in \mathbb{F}$ be defined by

$$f.g = \sum_{x \in A} f(x)g(x).$$

- By the basis property, there exists a function $h : A \to \mathbb{F}$ satisfying $h.a_i = 0$ for every $i \in [r]$ and $h.\delta_i = 1$ for every $i \in I$.
- Consider $h.T : A^2 \to \mathbb{F}$ defined by

$$(h.T)(y, z) = \sum_{x \in A} h(x)T(x, y, z).$$

The slice rank of a diagonal tensor

- From the slice rank decomposition,

$$(h.T)(y,z) = \sum_{j=1}^{s} c_j(y)(h.d_j)(z) + \sum_{k=1}^{t} e_k(z)(h.f_k)(y)$$

  for some functions $h.d_j : A \to \mathbb{F}$ and $h.f_k : A \to \mathbb{F}$. Hence $\mathrm{rk}(h.T) \leq s + t$.
- Meanwhile, from the fact that $T(x,y,z) \neq 0$ if and only if $x = y = z$,

$$(h.T)(y,z) = \sum_{a \in A} \tau_a 1_{y=z=a}$$

  with $\tau_a \neq 0$ for every $a \in I$, so $\mathrm{rk}(h.T) \geq |I| = |A| - r$.
- It follows that $r + s + t \geq |A|$, so the slice rank of T is at least $|A|$.

A slice rank upper bound

Proposition

The slice rank of the function $1_{x+y+z=0^n}$ is $O(C^n)$ for some $C < 3$.

A slice rank upper bound

- We write the identity

$$1_{x+y+z=0^n} = \prod_{i=1}^{n}(1-(x_i+y_i+z_i)^2).$$

- The right-hand side is a linear combination of monomials of the type

$$x_1^{a_1}\dots x_n^{a_n} y_1^{b_1}\dots y_n^{b_n} z_1^{c_1}\dots z_n^{c_n}$$

  satisfying the following conditions:
  - $a_1,\dots,a_n,b_1,\dots,b_n,c_1,\dots,c_n \in \{0,1,2\}$
  - $(a_1+\dots+a_n)+(b_1+\dots+b_n)+(c_1+\dots+c_n) \leq 2n.$

A slice rank upper bound

- By the pigeonhole principle, at least one of the three sums in the expression above is at most $2n/3$.
- So we can write
$$1_{x+y+z=0^n} = P_1 + P_2 + P_3,$$
where the monomials in P_1, P_2, P_3 are such that the first sum, second sum, third sum respectively are at most $2n/3$.

A slice rank upper bound

- We can write
$$P_1(x,y,z) = \sum_{\alpha} f_\alpha(x) g_\alpha(y,z)$$
where the sum is taken over all $\alpha = (a_1, \ldots, a_n) \in \{0,1,2\}^n$ with $a_1 + \cdots + a_n \leq 2n/3$.
- Let N be the number of such possibilities for α. We have established that the slice rank of $1_{x+y+z=0^n}$ is at most $3N$, and it now suffices to obtain an upper bound on N.
- We have
$$N = \sum_{u,v,w \geq 0, u+v+w=n, v+2w \leq 2n/3} \frac{n!}{u!v!w!}.$$

A slice rank upper bound

- By Stirling's formula, if $u = (p_u + o(1))n, v = (p_v + o(1))n, w = (p_w + o(1))n$ then

$$\frac{n!}{u!v!w!} = \exp(n(h(p_u, p_v, p_w) + o(1))),$$

  where $h(p_u, p_v, p_w)$ is the entropy of a random variable taking three values with probabilities p_u, p_v, p_w.
- We then have $N = \exp(n(M + o(1)))$, where M is the maximiser of $h(p_u, p_v, p_w)$ given

$$p_u, p_v, p_w \geq 0, p_u + p_v + p_w = 1, p_v + 2p_w \leq 2/3.$$

- We obtain $N = O(C^n)$ for $C = 2.756 < 3$.

Conclusion

Thank you for listening !