Lecture 1. The notions of rank on tensors and their history

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History of the ranks of tensors

- Thousands of years ago, the elimination algorithm for systems of linear equations was discovered.
- Around the 19th century, this algorithm was used to define the rank of a matrix and to develop the basic theory of linear algebra.
- During the 20th century, the tensor rank was defined by Hitchcock 1927, then rediscovered by Carroll, Chang, Harshman 1970 after which its use became widespread.
- During the last 10 years: new competing rank notions on tensors. Slice rank, partition rank, geometric rank, R-rank, local rank, etc.

Definitions of rank on tensors, 1/3

- An order-d tensor

$$T : [n_1] \times \cdots \times [n_d] \to \mathbb{F}$$

is the generalisation of a matrix to d dimensions instead of 2.

- For $d \geq 3$ there are several competing generalisations of rank. All of them however have the same structure: we first specify which tensors have rank at most 1, and then we say that the rank of a tensor T is the smallest nonnegative integer k such that we can write

$$T = T_1 + \cdots + T_k$$

with $T_1, \ldots, T_k$ tensors of rank at most 1.

Definitions of rank on tensors, 2/3

- Tensor rank: the rank 1 tensors are the products $a_1(x_1) \dots a_d(x_d)$.
- Slice rank, introduced by Tao (2016) in his reformulation of the cap-set breakthrough by Ellenberg-Gijswijt building on work of Croot-Lev-Pach: the rank 1 tensors are the products

$$a(x_1)b(x_2, \dots, x_d), a(x_2)b(x_1, x_3, \dots, x_d), \dots, a(x_d)b(x_1, \dots, x_{d-1}).$$

- Partition rank, defined by Naslund (2017) to obtain existence results on k-right corners: the rank 1 tensors are the products

$$a(x(I))b(x(J))$$

where $\{I, J\}$ is a non-trivial bipartition of $\{1, \dots, d\}$.

Definitions of rank on tensors, 3/3

- More generally, for R a non-empty family of partitions of $\{1, \ldots, d\}$ we can define the R-rank to be the notion of rank where the rank-1 tensors are the products

$$\prod_{I \in P} a_I(x(I))$$

for some $P \in R$.

- We can check that for every $d \geq 2$, the R-rank specialises to the tensor rank, to the slice rank, and to the partition rank, by taking respectively

$$R = \{\{\{1\}, \{2\}, \ldots, \{d\}\}\}$$
$$R = \{\{\{1\}, \{1\}^c\}, \{\{2\}, \{2\}^c\}, \ldots, \{\{d\}, \{d\}^c\}\}$$
$$R = \{\{I, J\} : \{I, J\} \text{ a bipartition of } [d] \text{ with } I, J \neq \emptyset\}.$$

Applications of the tensor rank

- Appears to be the oldest notion of rank on tensors.
- Heavily used in computational complexity theory.
- Connection to the question of the exponent of matrix multiplication.
- The tensor rank is known to be NP-hard to compute (and NP-complete over finite fields).
- In applied mathematics, expanding area of research devoted to algorithms that can find tensor rank decompositions or approximate tensor rank decompositions.

Birth of the slice rank

Problem

Let $A \subset \mathbb{F}_3^n$ such that the only $(x, y, z) \in A \times A \times A$ satisfying $x + y + z = 0$ are the triples (x, x, x). How large can A be?

- Trivial bound $|A| \leq 3^n$.
- Meshulam (1995): $|A| \leq O(3^n/n)$ using Fourier density-increment methods.
- Bateman and Katz (2012): $|A| \leq O(3^n/n^{1+a})$ for some $a > 0$.
- Croot-Lev-Pach and Ellenberg-Gijswijt (2016), both published in the Annals of Mathematics: $|A| \leq O(C^n)$ for some $C < 3$ (about 2.756).
- Proof shortly after reformulated in a more symmetric way by Tao in terms of a notion on tensors that was then called the *slice rank*.

A definition of rank that does not rely on decompositions

- Analytic rank. If T is an order-d tensor, then we can assign to it a d-linear form

$$m(y_1, \ldots, y_d) = \sum_{i_1,\ldots,i_d} T(i_1, \ldots, i_d) y_{1,i_1} \ldots y_{d,i_d}.$$

If $\mathbb{F}$ is finite, then we can define the analytic rank of T by

$$\mathrm{ar}\, T = -\log_{|\mathbb{F}|} \mathrm{bias}\, m$$

where

$$\mathrm{bias}\, m = \mathbb{E}_{y_1,\ldots,y_d \in \mathbb{F}^n} \chi(m(y_1, \ldots, y_d)) > 0$$

for some non-trivial character χ.

Partition rank versus analytic rank

- For matrices, $\text{ar}\, T = \text{rk}\, T = \text{pr}\, T$.
- If $\text{pr}\, T = 1$, then m splits as a product of two multilinear forms, so takes the value 0 with higher probability than usual.
- More generally $\text{ar}\, T \leq \text{pr}\, T$ (Lovett 2018).
- Can we conversely obtain upper bounds on $\text{pr}\, T$ in terms of $\text{ar}\, T$? Several authors have conjectured or asked whether $\text{pr}\, T \leq C(d)\, \text{ar}\, T$.
- Power bounds (Janzer 2019, Milićević 2019).
- Linear bounds for large fields (Moshkovitz and Cohen 2021).
- Quasi-linear bounds (Moshkovitz and Zhu 2022).

Examples of applications of the ranks of tensors

- Communication complexity
- Circuit complexity
- Quantum information theory
- Data compression
- Machine learning
- Network analysis
- Neuroscience

Despite numerous applications such as those above, the basic theory of these ranks is in its early stages of development.

Conclusion

Thank you for listening !