

Combinatorics, geometry, and topology of Bier spheres

Rade Živaljević
Mathematical Institute SASA, Belgrade, Serbia
rade@turing.mi.sanu.ac.rs

This is joint work with
Filip Jevtić, Marija Jelić Milutinović, Ivan Limonchenko,
and Marinko Timotijević

Each simplicial complex K (or a simple game $2^{[n]} \setminus K$) with n vertices (n players) is associated an $(n - 2)$ -dimensional, combinatorial sphere on (at most) $2n$ -vertices. This is the so called Bier sphere $Bier(K)$ (named after Thomas Bier [1]), formally defined as the deleted join $Bier(K) := K *_\delta K^\circ$ of K with its (combinatorial) Alexander dual K° . Bier spheres have been studied from the viewpoint of combinatorics (simplicial complexes), topology (polyhedral products, toric manifolds), convex polytopes (generalized permutahedra, algorithmic Steinitz problem), game theory (cooperative games), experimental mathematics (nonpolytopal spheres), combinatorial optimization (submodular functions), etc.

We present an overview of selected results from the multidisciplinary project *Bier-2016-2026*, initiated at ICERM in (2016), https://icerm.brown.edu/program/semester_program/sp-f16. One of the cornerstone results is the following theorem, which links Bier spheres with *braid arrangements, polymatroids, and generalized permutahedra*. This result and its ramifications, [6, 7, 8, 9, 10], pave the way for many applications, from polyhedral combinatorics to toric geometry and topology.

Theorem 1.([6, 7, 10]) *Each Bier sphere admits a canonical star-shaped geometric realization whose radial fan coarsens the braid arrangement fan. Moreover, this fan is unimodular in the lattice A_n^* , dual to the root lattice A_n .*

Acknowledgments. This work was partially supported by the Science Fund of the Republic of Serbia, Grant No. 7744592, Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics - MEGIC.

References

- [1] T. Bier. A remark on Alexander duality and the disjoint join, preprint (1992).
- [2] A. Björner, A. Paffenholz, J. Sjöstrand, G. M. Ziegler, Bier spheres and posets, *Disc. Comp. Geom.* **34** (2005), 71–86.
- [3] M. de Longueville, Bier spheres and barycentric subdivision, *J. Comb. Th., Series A* **105** (2004), 355–357.
- [4] J. Matoušek. *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*. Springer–Verlag, Berlin, 2003, 214pp.
- [5] F. D. Jevtić, M. Jelić, R. T. Živaljević. Cyclohedron and Kantorovich-Rubinstein polytopes, *Arnold Mathematical Journal*, 2018, Vol. 4, 87–112.
- [6] F. D. Jevtić, M. Timotijević, R. T. Živaljević, Polytopal Bier spheres and Kantorovich-Rubinstein polytopes of weighted cycles, *Disc. Comp. Geom.* **65** (2019), 1275–1286.
- [7] F. D. Jevtić and R. T. Živaljević. Bier spheres of extremal volume and generalized permutahedra. *Applicable Analysis and Discrete Mathematics*, 2022.
- [8] M. Timotijević, R. T. Živaljević, and F. D. Jevtić. Polytopality of simple games. *Experimental Mathematics*, Volume 34, 2025 – Issue 3.
- [9] I. Limonchenko, M. Timotijević, R. Živaljević, Bier spheres and the problem of toric non-quasitoric manifolds, *Rus. Math. Surveys* (2026), to appear.
- [10] I. Yu. Limonchenko, M. Timotijević, R. T. Živaljević, On a class of toric manifolds arising from simplicial complexes, preprint (2025); arXiv:2506.13547.