

## Geometry and Matrices: Extremal Theory (Minicourse)

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**LECTURE 1.** Clique number vs. independence number: Ramsey's theorem. Erdős-Hajnal conjecture. Intersection graphs of geometric objects, string graphs. Stronger versions of the Erdős-Hajnal property (strong, mighty). Enumeration of segment graphs, string graphs, pseudo-segment graphs.

**LECTURE 2.** Chromatic number vs. clique number. Chi-boundedness. Hadwiger's conjecture, Hajós conjecture.

**LECTURE 3.** Crossing numbers of graphs. Albertson's conjecture. Which crossing number is it, anyway? Three approaches to crossing numbers.

**LECTURE 4.** Unit distances, repeated distances. Incidences, cell decompositions. Erdős-Gyárfás problem. Szemerédi theorem on arithmetic progressions. Unit distances and (a) semialgebraic graphs, (b) rigidity, (c) 0-1 matrices.

**LECTURE 5.** Extremal problems in ordered graphs and 0-1 matrices. A variant of the Erdős-Stone-Simonovits theorem. Variants of the "almost linear conjecture" and special cases where they hold.

**LECTURE 6.** Edge-ordered graphs and why the "almost linear conjecture" holds for them.

**LECTURE 7.** Lower bound constructions.

**LECTURE 8.** The non-bipartite case: differences between the ordered and unordered settings. Results from (unordered) extremal graph theory that do carry over.

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### References

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