Eulerian Orientations

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This is joint work with Mikhail Isaev, Tejas Iyer and Rui-Ray Zhang.

An Eulerian orientation of a graph is an orientation of the edges such that, at each vertex, the number of incoming edges and the number of outgoing edges are equal. This implies that the degree of each vertex is even, which is easily seen to be also a sufficient condition.

This talk concerns the number $\mathrm{EO}(G)$ of Eulerian orientations. The quantity $\mathrm{eo}(G) = \frac{1}{n} \log \mathrm{EO}(G)$ is known as the residual entropy and can also be defined by a limiting process for infinite graphs with a periodic structure. A famous lower bound on $\mathrm{eo}(G)$ was introduced in 1967 by the physicist Elliot Lieb in his study of the behaviour of water ice.

We will consider three issues.

1. We will obtain precise estimates of $\mathrm{EO}(G)$ for graphs that are sufficiently dense and have sufficient expansion. This reveals an unexpected inverse relationship to the number of spanning trees that so far does not have a heuristic combinatorial explanation.
2. We will show that Lieb's bound is low by at most a constant and find an estimate for $\mathrm{eo}(G)$ in the case of regular graphs of low degree that outperforms previous estimates.
3. We will show that under weak conditions on the number of short cycles Lieb's estimate is asymptotically precise for regular graphs of increasing degree.