

## Spectral supersaturation for color-critical graphs

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A graph is *color-critical* if it contains an edge whose deletion reduces its chromatic number. This class of graphs, including cliques and odd cycles, plays a central role in extremal graph theory. In this paper, following an influential line of research initiated by Bollobás–Nikiforov, we study the spectral supersaturation problem for color-critical graphs. Let  $T_{n,r}$  be the  $r$ -partite Turán graph, let  $\mathcal{T}_{n,r,q}$  denote the family of graphs obtained from  $T_{n,r}$  by adding  $q$  edges, and let  $\lambda(G)$  be the spectral radius of a graph  $G$ . We first prove that for any color-critical graph  $F$  with chromatic number  $r+1$ , there exists  $\delta_F > 0$  such that for sufficiently large  $n$  and all  $1 \leq q \leq \delta_F \sqrt{n}$ , any  $n$ -vertex graph  $G$  with  $\lambda(G) \geq \min_{T \in \mathcal{T}_{n,r,q}} \lambda(T)$  contains at least  $q \cdot c(n, F)$  copies of  $F$ , where  $c(n, F)$  denotes the minimum number of copies of  $F$  created by adding a single edge to  $T_{n,r}$ ; moreover, any extremal graph  $G$  must belong to  $\mathcal{T}_{n,r,q}$ . Next, we prove a spectral supersaturation result for the analogous condition  $\lambda(G) \geq \max_{T \in \mathcal{T}_{n,r,q}} \lambda(T)$ , valid for all  $1 \leq q \leq \delta_F n$ . Together, these results provide a complete resolution to a problem proposed by Ning–Zhai, and establish a spectral counterpart to the well-known results of Mubayi and Pikhurko–Yilma in the extremal supersaturation setting. A notable feature of our first result is that the restriction  $q = O(\sqrt{n})$  is tight up to a constant factor, in contrast to the linear bounds provided by other settings discussed above. As applications, we extend a result of Liu–Mubayi, and solve a related conjecture by Li–Lu–Peng. Our proof is based on a novel spectral incremental technique for graphs close to the Turán graph  $T_{n,r}$ , which may be of independent interest.