From lattice covering numbers to the ranks of tensors

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Description

Tensors are higher-dimensional generalisations of matrices. The matrix rank has been generalised to several competing notions of rank on tensors that we will begin by presenting, paying particular attention to the slice rank and its role in the resolution of the cap-set problem. In a second part, after studying several basic properties of the covering numbers of lattices we will establish a result of Sawin and Tao providing an interpretation of the slice rank in terms of lattice coverings in the special case of вЂњsparseвЂќ tensors, which reduces the study of the former to the latter in that special case. In a third part, we will use the linear groups of transformations on tensors to explain why there is already a major gap in difficulty between the study of basic properties of the ranks of tensors between order-2 tensors (matrices) and order-3 tensors, and then discuss which properties can or cannot reasonably be expected to hold for the ranks of tensors. In a final part, we will discuss proofs of some of these properties, highlighting the guidance provided by lattice covering numbers even when the connection of Sawin and Tao does not apply.

Plan for the 8 lectures

1: The notions of rank on tensors and their history

Unlike in the case of matrices, there is no single canonical notion of rank of tensors. Rather, the most useful notion of rank depends on the context and application that one has in mind, and over the past decade alone many notions have seen the light of day, such as the slice rank [?] and partition rank [?], and more recently the geometric rank [?] and local rank [?]. This lecture focuses on a history of these notions and of their motivations and applications, emphasizing the major similarities and differences in the definitions.

2: The slice rank of tensors and the solution of the cap-set problem

This lecture focuses on the origin of the slice rank. The cap-set problem in number theory asks for the size of the largest subset $A \subset \mathbb{F}_3^n$ such that $A \times A \times A$ contains no solutions to $x + y + z = 0$ aside from the trivial solutions $x = y = z$. A breakthrough from the 2016 papers of Croot-Lev-Pach [?] and Ellenberg-Gijswijt [?] (both published in Ann. Math.) shows that $|A|/3^n$ must be exponentially small in n. We will present their proof reformulated in terms of the slice rank, that Tao defined afterwards [?] in order to write the proof while treating variables more symmetrically.

3: Lattice coverings: basic questions, results and proofs

This lecture introduces a second main object which will play an important role throughout the course: the covering number of a lattice subset, defined to be the smallest number of sets of some prescribed type (for instance, slices) that suffice to cover the lattice subset. We will prove some simple properties of lattice covering numbers [?]: for instance, must some small restriction of the lattice have the same covering number as the original lattice, and how much flexibility is there in choosing a lattice covering with minimal size?

4: A connection between the slice covering number and the slice rank

This lecture establishes a characterisation of the slice rank of a tensor by Sawin and Tao [?] in the special case where the support of the tensor is contained in an antichain: in this special case, the slice rank of the tensor is equal to the smallest number of slices that suffices to cover its support. From this connection it follows that in this special case, proving basic results on the slice rank reduces to proving the analogous results on the slice covering number.

5: The linear group of transformations on tensors, and the subrank

This lecture provides fundamental reasons why the study of rank notions on tensors is much more difficult than the study of the matrix rank. As the size of the group of base change transformations is considerably smaller than the set of order-d tensors on $[n]^d$ for $d \geq 3$, it is then no longer possible to reduce all tensors to вЂњsimpleвЂќ normal forms purely with base change transformations as could be done in the matrix case, although the idea still leads to the definition of another notion of rank on tensors called the subrank.

6: Formulations of basic properties of the ranks of tensors

This lecture formulates statements aiming to extend basic properties of matrix rank to the ranks of tensors, for instance the property that every rank-k matrix contains some $k \times k$ rank-k submatrix and the property that every matrix only has one minimal-length rank decomposition up to a change of basis. A theme arises repeatedly: the naive extension to the ranks of tensors of the property that held for matrix rank fails in a major way, but may nonetheless be rectified to a property which (in addition to being true) is not too complicated to state and in a spirit very close to that of the original property.

7: Proving the extension of the submatrix property to tensors

This lecture focuses on the proof [?] of the property that if a tensor has high rank then it must necessarily contain a subtensor with bounded size that nonetheless still has high rank, establishing it in some special cases involving ideas that are representative of the proof in the general case. The many analogies with the setting of lattice covering numbers [?] are stressed: even if the ideas from that setting do not by themselves suffice to obtain the proof for the ranks of tensors they lead to several central ideas in that proof, most notably a sufficient condition guaranteeing that a tensor has high rank and a sufficient condition guaranteeing that two notions of rank are вЂњequivalentвЂќ on some tensor.

8: Proving the boundedness for optimal slice rank decompositions

This lecture focuses on the proof [?] of the property that if an order-d tensor has slice rank k over a finite field $\mathbb{F}$, then up to a natural class of transformations (that is indispensable in such a statement) it may only have a number of minimal-length slice rank decompositions that is bounded in terms of $d, k, |\mathbb{F}|$. This concludes the course.

References

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