Ample sets in Cartesian products

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The main goal of the talk is to define ample sets in Cartesian products and to present their main characterizations and properties. First, we will briefly recall the definition and the properties of ample sets in hypercubes. In this case, ample sets were defined by A. Dress (1995) by a sandwich inequality and they turn out to be equivalent to lopsided sets of J. Lawrence (1983), to simple sets of D. Wiedemann (1986), and to extremal sets of B. Bollobàs and A.J. Ratcliffe (1995).

We consider subsets S of Cartesian products $U := U_1 \times \ldots \times U_m$ of nonempty finite sets $U_1, \ldots, U_m$. To define ampleness in this general setting, we apply to Cartesian products the *shattering→strong shattering principle*, which is one of the equivalent definitions of classical ampleness. In hypercubes $\{0,1\}^E$, this principle is asserting that whenever a hypercube $\{0,1\}^Y$ with $Y \subseteq E$ is shattered by S, then S contains a copy of $\{0,1\}^Y$ (i.e., it is strongly shattered by S). In case of Cartesian products, we apply the shattering→strong shattering principle to minor-subproducts of U. Each *minor subproduct* $M = M(\Lambda)$ of U is associated to a generalized partition $\Lambda = (\alpha_1, \ldots, \alpha_m)$, where each α_i is a partition of U_i. Then $M(\Lambda) = M_1 \times \ldots \times M_m$, where each M_i is the minor of U_i obtaining by contracting each block of the partition α_i into a single vertex. The generalized partition $\Lambda = (\alpha_1, \ldots, \alpha_m)$ defines a partition $\mathcal{B}(\Lambda)$ of U into boxes, which are Cartesian products of the blocks from different partitions $\alpha_i, i = 1, \ldots, m$ of Λ. Then, alternatively $M(\Lambda)$ is obtained from U by contracting each box of the partition $\mathcal{B}(\Lambda)$ into a single vertex. A minor-subpriduct $M(\Lambda)$ is *shattered* by S if S intersects each box of $\mathcal{B}(\Lambda)$ and M is *strongly shattered* by S if S contains a copy of $M(\Lambda)$. A set $S \subseteq U$ is called *ample* whenever a minor-subproduct $M(\Lambda)$ is shattered by S, then $M(\Lambda)$ is strongly shattered by S. In an analogous way, we can define the operations of *projection* S_M and *strong projection* of a set $S \subseteq U$ on a minor-subproduct M.

We show that the main characterizations (but not all) of ample sets of hypercubes generalize to ample sets of arbitrary Cartesian products. We will also present some other properties of ample sets: push-downs and corner peelings, Euler characteristic and contractibility of their box/prism complexes. Here is a list of main characterizations of ample sets:

Theorem 1. *For a set $S \subseteq U = U_1 \times \ldots \times U_m$, the following conditions are equivalent:*

1. *S is ample;*
2. *S^M is isometric for all minor-subproducts M (superisometricity);*
3. *S is box-superisometric;*
4. *S^M is connected for all minor-subproducts M (superconnectivity);*
5. *$(S^M)_{M'} = (S_{M'})^M$ for all minor-subproducts M, M' with disjoint supports (commutativity);*
6. *the complement $S^* = U - S$ is ample.*
7. *S is isometric and both S^e and S_e are ample for some elementary minor-subproduct $e = \{a, b\}$;*
8. *S is connected and S^e is ample for every elementary minor-subproduct e;*
9. *$S \cap [u, v]$ is ample in $[u, v]$ for all $u, v \in S$.*