Digraphs with non-diagonalizable adjacency matrix

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The fact that the adjacency matrix of every finite graph is diagonalizable plays a fundamental role in spectral graph theory. Since this fact does not hold in general for digraphs, it is natural to ask whether it holds for digraphs with certain level of symmetry. Interest on this question dates back to early 1980s, when P. J. Cameron [1] asked for the existence of arc-transitive digraphs with non-diagonalizable adjacency matrix. This was answered in the affirmative by L. Babai [2] in 1985. Then Babai [2] posed the open problem of constructing a 2-arc-transitive digraph and a vertex-primitive digraph whose adjacency matrices are not diagonalizable. In this talk, we solve Babai's problem by constructing infinite families of digraphs with the required properties.

To build an infinite family of digraphs from an existing one, we use the *tensor product* $\Gamma \times \Sigma$ of digraphs Γ and Σ, where $\Gamma \times \Sigma$ is the digraph with vertex set $V(\Gamma) \times V(\Sigma)$ such that $(u_1, v_1) \to (u_2, v_2)$ if and only if $u_1 \to u_2$ in Γ and $v_1 \to v_2$ in Σ. For an integer $n \geq 1$, denote by $\Gamma^{\times n}$ the tensor product of n copies of digraph Γ. Our main result gives infinite families of non-diagonalizable s-arc-transitive digraphs and non-diagonalizable vertex-primitive digraphs. The basic digraphs in these two families are as follows.

Construction 1. For each integer $s \geq 2$, let $a_s = (2s-1, 2s)(4s-1, 4s) \in \mathrm{Sym}_{4s}$, let $b_s = (1, 3, 5, \ldots, 4s-1, 2, 4, 6, \ldots, 4s) \in \mathrm{Sym}_{4s}$, let $R_s = \langle a_s, b_s \rangle$ be the group generated by a_s and b_s, and let $\Gamma_s = \mathrm{Cay}(R_s, \{a_s b_s, b_s\})$.

Construction 2. Let $R = \langle a, b \mid a^7 = b^3 = 1,\, b^{-1}ab = a^2 \rangle \times \langle c, d \mid c^7 = d^3 = 1,\, d^{-1}cd = c^2 \rangle$, let γ be the automorphism of R interchanging a with c and b with d, let

$$S = (S_1 \cup S_1^{-1})(S_3 \cup S_3^{-1})^\gamma \cup (S_3 \cup S_3^{-1})(S_1 \cup S_1^{-1})^\gamma \cup S_1 S_2^\gamma \cup S_2 S_1^\gamma \cup S_1^{-1} S_4^\gamma \cup S_4 (S_1^{-1})^\gamma,$$

where

$$S_1 = \{a,\, a^5,\, a^6 b,\, a^6 b^2\}, \quad S_2 = \{ab,\, (ab)^{-1}\},$$
$$S_3 = \{a^3,\, b,\, ab^2,\, a^4 b^2\}, \quad S_4 = \{a^2 b,\, (a^2 b)^{-1}\},$$

and let $\Sigma = \mathrm{Cay}(R, S)$.

Our main result is as follows.

Theorem 3. For all positive integers n and $s \geq 2$, for the digraph Γ_s in Construction 1 and for the digraph Σ in Construction 2, the digraphs $\Gamma_s^{\times n}$ and $\Sigma^{\times n}$ satisfy the following:

(a) $\Gamma_s^{\times n}$ is s-arc-transitive;

(b) $\Sigma^{\times n}$ is vertex-primitive;

(c) $\Gamma_s^{\times n}$ and $\Sigma^{\times n}$ are non-diagonalizable.

References

[1] P.J. Cameron, Automorphism groups of graphs, in Selected Topics in Graph Theory Vol. 2, (L.W. Beineke and R.J. Wilson. eds.), Academic Press, New York, (1983) 89–127.

[2] L. Babai, Arc transitive covering digraphs and their eigenvalues, J. Graph Theory, 9 (1985) 363–370.