New combinatorial problems from the Hermitian Sum-of-Squares conjecture

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This is joint work with Anyuan Tian and Yaokun Wu

Let $n, s \geq 2$ be integers, and let $(\mathbb{Z}^n_{\geq 0})_s$ be the set of n-tuples of nonnegative integers summing to s. For each $i = 1, \dots, n$, denote by $\mathbf{e}_i$ the length-n $(0,1)$-vector with a 1 in the i-th coordinate and 0 elsewhere. For any subset $Z \subseteq (\mathbb{Z}^n_{\geq 0})_s$, define $\partial Z = \{z + \mathbf{e}_i \colon z \in Z, 1 \leq i \leq n\}$. Set $\Gamma_{n,s} = \{(X, Y) \colon X, Y \subseteq (\mathbb{Z}^n_{\geq 0})_s, X \cap Y = \emptyset, \partial X \supseteq \partial Y, Y \neq \emptyset\}$ and $\mathrm{p}_n(s) = \min\{|\partial X \setminus \partial Y| \colon (X, Y) \in \Gamma_{n,s}\}$. We prove that

$$\mathrm{p}_n(s) = \begin{cases} 8, & \text{if } n = 4, \\ \frac{(n-1)(n+2)}{2}, & \text{if } n \neq 4. \end{cases}$$

This result is motivated by the Hermitian Sum-of-Squares (SOS) conjecture in several complex variables proposed by Ebenfelt [?, Conjecture 1.2]. Consequences of our combinatorial result include that the SOS conjecture holds in the monomial case, and that a related conjecture raised by Wang, Yue, and Zhou [?, p. 9] is valid. If time permits, we will also report some combinatorial investigations of the general SOS conjecture.

References

[1] Peter Ebenfelt, On the HJY gap conjecture in CR geometry vs. the SOS conjecture for polynomials, in: Analysis and Geometry in Several Complex Variables, Amer. Math. Soc., Providence, RI, 2017, 125–135.

[2] Zhiwei Wang, Chenlong Yue, Xiangyu Zhou, A Newton-Okounkov body viewpoint on the SOS conjecture. arXiv:2512.07133 (2025).