Amorphic association schemes and beyond

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This is joint work with Edwin van Dam and Jack Koolen

Let $\mathcal{R}$ be a d-class association scheme with relations $A_0(=I), A_1, \ldots, A_d$. Let π be a partition of $\{1, \ldots, d\}$ with blocks $\pi(1), \ldots, \pi(d')$. We say that π gives rise to a fusion scheme $\mathcal{R}_\pi$ of $\mathcal{R}$ if the configuration $\mathcal{R}_\pi$ with relations $A_0, A_{\pi(1)}, \ldots, A_{\pi(d')}$ — where $A_{\pi(j)} = \sum_{i\in\pi(j)} A_i$ — is an association scheme. We call $\mathcal{R}$ *amorphic* if every partition π of $\{1, \ldots, d\}$ gives rise to a fusion scheme of $\mathcal{R}$.

Ivanov showed that all nontrivial relations (relations other than A_0) in an amorphic d-class association scheme with $d \geq 3$ are strongly regular of (negative) Latin square type. He conjectured that an association scheme is amorphic if every relation is strongly regular. This conjecture was later disproved, leading to many interesting constructions of amorphic or nonamorphic association schemes.

In this talk, I will present our characterizations of amorphic association schemes and our constructions of almost amorphic association schemes. Our main results are as follows.

In [1], in terms of fusing pairs of relations, we give three equivalent conditions for $\mathcal{R}$ to be amorphic:

- Every pair $\{A_i, A_j\}$ $(1 \leq i < j \leq d)$ fuses;
- For $d \geq 3$, the fusing-relations graph of $\mathcal{R}$ is connected but not a path;
- At most one nontrivial relation fails to be a strongly regular graph of (negative) Latin square type.

The *fusing-relations graph* of $\mathcal{R}$ is defined as the graph on the vertex set $\{A_1, \ldots, A_d\}$ where A_i and A_j are joined by an edge if fusing them yields a fusion scheme.

In [2], in terms of fusing triples of relations, we give two equivalent conditions for $\mathcal{R}$ to be amorphic:

- For $d \geq 5$, every triple $\{A_i, A_j, A_k\}$ $(1 \leq i < j < k \leq d)$ fuses;
- For $d \geq 5$, the fusing-relations 3-hypergraph of $\mathcal{R}$ contains two 3-sunflowers.

The *fusing-relations* 3*-hypergraph* of $\mathcal{R}$ is defined as the 3-uniform hypergraph on the vertex set $\{A_1, \ldots, A_d\}$ where $\{A_i, A_j, A_k\}$ is an edge if and only if fusing them yields a fusion scheme. A 3*-sunflower* is a 3-hypergraph such that its edges cover all vertices and have exactly two vertices in common.

In [3], we prove that for any $d \geq 2$, there exist nonamorphic d-class association schemes in which exactly $d-2$ of the nontrivial relations are strongly regular of Latin square type.

References

[1] E. R. van Dam, J. H. Koolen, Y. Xiong, Characterizations of amorphic schemes and fusions of pairs, *J. Combin. Theory Ser. A* **215**, 106045 (2025).

[2] Y. Xiong, Characterizations of amorphic association schemes in terms of fusing triples, 2026, arXiv: 2604.27360.

[3] E. R. van Dam, J. H. Koolen, Y. Xiong, Almost amorphic association schemes, 2026, arXiv: 2604.06538.