Converse CaseyXs theorem in hyperbolic geometry

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This is joint work with Nikolay Abrosimov

In 1881 Irish mathematician John Casey generalized PtolemyXs theorem in the following way (see [?], p. 103).

Casey's theorem *Let circles O_1, O_2, O_3, O_4 on a plane touch given circle O in vertices p_1, p_2, p_3, p_4 of a convex quadrilateral. Denote by t_{ij} the length of a common tangent of the circles O_i and O_j. If O separates O_i and O_j then the internal tangent should be taken as t_{ij} else the external tangent should be taken. In both cases we assume that the tangents are exist. Then*

$$t_{12}t_{34} + t_{23}t_{14} = t_{13}t_{24}.$$

The converse to CaseyXs theorem in $\mathbb{E}^2$ was proved much later by different authors, and with different kinds of restrictions (see [?]).

Casey's converse theorem *Given four circles O_1, O_2, O_3, O_4 on a plane and t_{ij} denotes a length of common tangent of circles O_i and O_j. If the orientations of O_i and O_j coincide then t_{ij} is external tangent, otherwise t_{ij} is internal one. Then if one of three numbers $t_{12} \cdot t_{34}, t_{23} \cdot t_{14}, t_{13} \cdot t_{24}$ is equal to the sum of two others, then the circles O_1, O_2, O_3, O_4 either pass through one point or touch a circle O or touch one straight line P.*

In our paper [?] we obtained a hyperbolic version of Casey's direct theorem, while the converse theorem remained unproven in $\mathbb{H}^2$. During the talk we will give the proof of Casey's converse theorem on the Lobachevsky plane.

Casey's converse theorem in $\mathbb{H}^2$ *Given four circles O_1, O_2, O_3, O_4 on the hyperbolic plane and T_{ij} denotes a length of common tangent of circles O_i and O_j. If the orientations of O_i and O_j coincide then T_{ij} is external tangent, otherwise T_{ij} is internal one. Then if one of three numbers*

$$\sinh t_{12} \cdot \sinh t_{34}, \quad \sinh t_{23} \cdot \sinh t_{14}, \quad \sinh t_{13} \cdot \sinh t_{24}$$

is equal to the sum of two others, then the circles O_1, O_2, O_3, O_4 either pass through one point or touch a curve O of constant geodesic curvature (that is a circle, horocycle or one branch of equidistant.

References

[1] J. Casey, *A sequel to the first six books of the Elements of Euclid, containing an easy introduction to modern geometry, with numerous examples, 5th. ed.*, Hodges, Figgis and Co., Dublin (1888).

[2] R.A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Houghton Mifflin, Boston (1929).

[3] N. V. Abrosimov, L. A. Mikaiylova, CaseyXs theorem in hyperbolic geometry. *Siberian Electronic Mathematical Reports* **12** (2015) 354–360.