Skew braces with no proper left ideals

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Skew brace is a group-like and ring-like algebraic structure that was introduced in [2, 4] as a tool to study set-theoretic solutions to the Yang–Baxter equation. Recall that a *skew brace* is a set $A = (A, \cdot, \circ)$ equipped with two group operations $\cdot$ and $\circ$ such that left-distributive-like relation

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$$

holds for all $a, b, c \in A$, where a^{-1} denotes the inverse of a in $(A, \cdot)$. It is easy to see that $(A, \cdot)$ and $(A, \circ)$ share the same identity element 1. Moreover, it is not difficult to show that

$$\lambda_a : (A, \cdot) \longrightarrow (A, \cdot); \quad \lambda_a(b) = a^{-1} \cdot (a \circ b)$$

is an isomorphism for every $a \in A$, and the map

$$\lambda : (A, \circ) \longrightarrow \mathrm{Aut}(A, \cdot); \quad a \mapsto \lambda_a$$

is a homomorphism. A subgroup I of $(A, \cdot)$ is said to be a *left ideal* of A if $\lambda_a(I) \subseteq I$ holds for all $a \in A$. For example, the characteristic subgroups of $(A, \cdot)$ are clearly left ideals of A. Left ideals are ubiquitous in the study of skew braces and play a very important role. Let us say that A is *left-simple* if $A \neq 1$ and A has no left ideals other than 1 and A. If A is left-simple, then clearly $(A, \cdot)$ has to be characteristically simple, so we have $(A, \cdot) \simeq T^n$ for some $n \in \mathbb{N}$ and simple group T when A is finite.

Our main result is the following partial classification of finite left-simple skew braces.

Theorem 1. *Let $A = (A, \cdot, \circ)$ be a finite skew brace such that $(A, \cdot) \simeq \mathbb{F}_p^n$ for some prime p and $n \in \mathbb{N}$. Then A is left-simple if and only if $n = 1$, in which case A is trivial, i.e. $a \circ b = a \cdot b$ for all $a, b \in A$.*

Theorem 2. *Let $A = (A, \cdot, \circ)$ be a finite skew brace such that $(A, \cdot) \simeq T^n$ for some non-abelian simple group T and $n \in \mathbb{N}$. It is well-known that $\mathrm{Aut}(T^n) = \mathrm{Aut}(T) \wr S_n = \mathrm{Aut}(T)^n \rtimes S_n$, where S_n denotes the symmetric group on n letters.*

(*a*) *For $n = 1$, we have A is left-simple if and only if A is almost trivial, i.e. $a \circ b = b \cdot a$ for all $a, b \in A$.*

(*b*) *For $n \geq 2$, if A is left-simple, then $(A, \circ)$ is isomorphic to some subgroup of $\mathrm{Aut}(T^n)$ whose projection onto S_n is a transitive subgroup, and $\mathrm{Im}(\lambda)$ intersects trivially with $\mathrm{Inn}(A, \cdot)$.*

The proof of Theorem 1 uses modular representation theory and is fairly simple. On the other hand, the proof of Theorem 2(a) relies on the classification of factorizations of finite almost simple groups with a solvable factor that is given in [3].

Remark. The motivation of this research originated from Hopf–Galois theory on finite Galois extensions. More specifically, it follows from [5] that finite left-simple skew braces correspond to minimal Hopf–Galois structures on finite Galois extensions, where *minimal* is defined as in [1].

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