Plans theorem for a cone over sandwich graphs

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This talk considers a family of graphs called sandwich graphs. The structure of the Jacobian for a cone over a sandwich graph is studied. Namely, a result analogous to Plans theorem for knots is established for this mathematical object. Classical Plans theorem states that the first homology group of an odd-fold cyclic covering of the sphere $\mathbb{S}^3$, branched over a knot, is the direct sum of two copies of some Abelian group. A similar result holds for the first homology group of an even-fold cyclic covering factored by the reduced homology group of a two-fold covering. Special cases of sandwich graphs are the I-graph and the generalized Petersen graph (see, for example, [2, 3, 5]), as well as the generalized prism [1].

Consider a graph consisting of a single vertex v and having no edges. A *cone over a graph* G is a graph $\widehat{G} = G \star \{v\}$ whose set of vertices and edges is $V(\widehat{G}) = V(G) \cup v$ and $E(\widehat{G}) = E(G) \cup \{\{w, v\},\, w \in V(G)\}$. The cone $\widehat{G} = G \star \{v\}$ over G is a discrete analogue of n-fold cyclic branched covering over the knot from Plans theorem. Let us introduce the concept of a sandwich graph.

Let $n \in \mathbb{N}$, consider a finite set of integers $1 \leqslant s_1 < s_2 < \ldots < s_k \leqslant \frac{n}{2}$. A graph $G = C_n(s_1, s_2, \ldots, s_k)$ on n vertices is called *a circulant graph* if each of its i-th vertices is adjacent to vertices $i \pm s_1, \ldots, i \pm s_k$ modulo n. Let P_m be a path-graph with vertices v_j. Consider circulant graphs $G_j = C_n(s_{j1}, s_{j2}, \ldots, s_{jk_j})$ with n vertices, $j = 1, 2, \ldots, m$, and define the *sandwich graph* $SG_n = SG_n(G_1, G_2, \ldots, G_m)$ as a graph with vertex set $V(SG_n) = \{(\ell, v_j) \,|\, \ell = 1, 2, \ldots n,\ j = 1, 2, \ldots, m\}$ in which each vertex (ℓ, v_j) is adjacent to the vertices $(\ell \pm s_{ij1}, v_i), (\ell \pm s_{j2}, v_j), \ldots, (\ell \pm s_{jk_j}, v_j)$ modulo n, and for all $\ell = 1, 2, \ldots n$ the vertices (ℓ, v_1), $(\ell, v_2), \ldots, (\ell, v_m)$ form a path graph P_m.

Denote a cone over SG_n as $\widehat{SG}_n$. For each graph $\widehat{SG}_n$, a monic Laurent polynomial $P(t)$ is defined. The structure of $P(t)$ and its coefficients are described in the article [4]. Note $P(t) \in \mathbb{Z}[t, t^{-1}]$ is palindromic, hence it can be transformed using the Chebyshev transform into a composition of functions

$$P(t) = Q \circ h(t) = a_0 + 2\sum_{k=0}^{m} a_k \mathcal{T}_k\left(\frac{h(t)}{2}\right), \quad \text{where} \quad Q(t) = a_0 + 2\sum_{k=0}^{m} a_k \mathcal{T}_k\left(\frac{t}{2}\right),$$

and $h(t) = t + t^{-1}$. The companion matrices C_P and C_Q are well defined. Here $\mathcal{T}_s(t)$, $\mathcal{U}_s(t)$ and $\mathcal{W}_s(t)$ are Chebyshev polynomials of the first, second and fourth kinds, respectively.

Theorem. *For the Jacobian of the cone* $\widehat{SG}_n$ *the following statements are valid.*

1. *If* n *is odd, then there exists an epimorphism* $\varphi : Jac(\widehat{SG}_{2r+1}) \to Jac(\widehat{SG}_1)$ *whose kernel splits into direct sum of two copies of the Abelian group represented by the matrix* $\mathcal{W}_r\,(C_Q/2)$.
2. *If* n *is even, then there exists an epimorphism* $\varphi : Jac(\widehat{SG}_{2r}) \to Jac(\widehat{SG}_2)$ *whose kernel splits into direct sum of two copies of the Abelian group represented by the matrix* $\mathcal{U}_{r-1}\,(C_Q/2)$.

Acknowledgments. The work is supported by the Mathematical Center in Akademgorodok under the agreement No. 075-15-2025-348 with the Ministry of Science and Higher Education of the Russian Federation.

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