

**$q$ -deformation of graphical arrangement and its generalizations**

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We introduce two new classes of hyperplane arrangements inspired by graphic arrangements and their  $q$ -deformations. Building on the analogy between the braid arrangement and the arrangement of all hyperplanes over a finite field, we extend the notion of  $q$ -deformation from graphs to simplicial complexes by replacing cliques with faces. We also study a special 1-dimensional setting and discuss characteristic polynomials, freeness, and extensions to fields containing primitive roots. This talk is based on [1].

In [2], the authors introduce a kind of arrangement on graphs called  $q$ -deformation, defined as follows.

**Definition 1.** Let  $G = ([\ell], E)$  be a simple graph on  $\ell$  vertices,  $C(G)$  denotes the set of cliques of the graph. The  $q$ -deformation of graphic arrangement  $\mathcal{A}_G$  is defined by

$$\mathcal{A}_G^q = \bigcup_{\{i_1, \dots, i_r\} \in C(G)} \ker(a_{i_1}x_{i_1} + a_{i_2}x_{i_2} + \dots + a_{i_r}x_{i_r}) \mid a_{i_j} \in \mathbb{F}_q^\times \text{ for all } j.$$

And a certain limit  $q \rightarrow 1$  is expected to recover the chromatic polynomial as follows.

**Conjecture 2.** The characteristic polynomial of the  $q$ -deformation  $\chi(\mathcal{A}_G^q, t)$  is a polynomial in  $q$  and  $t$ , such that

$$\lim_{q \rightarrow 1} \frac{\chi(\mathcal{A}_G^q, q^s)}{(q-1)^\ell} = \chi(G, s),$$

where  $\chi(G, t)$  denotes the chromatic polynomial of the graph  $G$ .

In [1], we prove that triangle-free graphs satisfy this conjecture by generalizing  $q$ -deformations to simplicial complexes and extend the conjecture to the case of simplicial complexes. By restricting simplicial complexes to 1-dimensional case, we prove that all 1-dimensional simplicial complexes satisfy the generalized conjecture.

Finally we extend the field from  $\mathbb{F}_q$  to all fields with primitive  $r$ -th root of unity and call this kind of arrangement **graphic monomial arrangement** since it shows close relationship with monomial arrangements based on reflection groups.

**Definition 3.** Let  $G = ([\ell], E)$  be a graph,  $z$  be a primitive  $r$ -th root of unity in the field. Define the **graphic monomial arrangement**  $\mathcal{M}(G, r)$  by

$$\mathcal{M}(G, r) = \{\{x_i - z^k x_j = 0\} \mid (i, j) \in E, 1 \leq k \leq r\} \cup \{\{x_i = 0\} \mid i = 1, \dots, \ell\}.$$

We show that graphic monomial arrangements satisfy the generalized conjecture and show the equivalence between chordality of the underlying graph and freeness of the corresponding graphic monomial arrangement, following the proof of  $q$ -deformations and graphic arrangements. We give an explicit free basis of this arrangement using combinations of determinants of Vandermonde matrices.

**References**

- [1] T. Nian, A generalization of  $q$ -deformation of graphic arrangements to simplicial complexes. arXiv:2601.06546 (2026).
- [2] T. Nian, S. Tsujie, R. Uchiyumi, M. Yoshinaga.  $q$ -Deformation of Graphic Arrangements, *Séminaire Lotharingien de Combinatoire 93B.149* (2025).