Laplace polynomials for the Moebius strip and Klein bottle

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Riemann surfaces endowed with Euclidean geometry have special properties and are unique exceptions in the theory of Riemann surfaces. The list of these surfaces is finite and well known. These include the torus, cylinder, Moebius strip, and Klein bottle. The fundamental groups of these surfaces are non-Hopfian and admit nontrivial endomorphisms. This circumstance leads to the possibility of nontrivial coverings of the surface over itself and other surfaces.

This talk is devoted to the properties of the Laplace operator defined on discrete versions of the above-mentioned Euclidean surfaces. Discrete analogues of these surfaces are presented as rectangular lattices, with suitable identification of the sides. The goal is to study the structure of the characteristic polynomial of the discrete Laplace operator for such graphs.

Consider the discrete Moebius strip $M_{m,n}$ obtained by identification of the sides of $m \times n$ rectangular lattice. Then the following theorem holds.

Theorem 1. *The characteristic polynomial of discrete Moebius strip $M_{m,n}$ is given by the formula*

$$\chi_{M_{m,n}}(\lambda) = \prod_{j=1}^{n} (2T_m(2 - \frac{\lambda}{2} - \cos\frac{\pi\,(j-1)}{n}) + (-1)^j 2)$$

where $T_n(w) = \cos(n \arccos(w))$ is the Chebyshev polynomial of the first kind.

Similar theorems proved for the discrete Klein bottle and other surfaces. Non-Hopfian property of Euclidean surfaces translates into curious property of theirs respective discrete variants. In particular, $\chi_{M_{m,n}}(\lambda)$ divides $\chi_{M_{pm,qn}}(\lambda)$ if $p+q$ is even.

Generally, for any discrete Euclidean surface obtained from $m \times n$ rectangular lattice, its characteristic polynomial $\chi_{m,n}(\lambda)$ divides polynomial $\chi_{pm,qn}(\lambda)$ for natural numbers p and q.

The related results are partially published in papers [**?**, **?**]:

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