

The Dunkl-Watanabe Duality

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Let \tilde{X} denote the set of all subspaces of the N -dimensional vector space \mathbb{F}_q^N over the finite field \mathbb{F}_q of q elements. The general linear group $GL(N, q)$ acts on \tilde{X} naturally. Fix a D -dimensional subspace x_0 from \tilde{X} and let H be the stabilizer in $GL(N, q)$ of x_0 .

Let $M_{\tilde{X}}(\mathbb{C})$ denote the full matrix algebra over the complex number field \mathbb{C} whose rows and columns are indexed by elements of \tilde{X} . Let $M_{\tilde{X}}^H(\mathbb{C})$ denote the centralizer algebra of H , i.e., the set of elements of $M_{\tilde{X}}(\mathbb{C})$ that commute with the action of H on \tilde{X} .

In this talk, we show $M_{\tilde{X}}^H(\mathbb{C}) = \mathcal{H}$, where \mathcal{H} is the algebra introduced by Watanabe in [3]. We call the algebra \mathcal{H} the Watanabe algebra, and the relation $M_{\tilde{X}}^H(\mathbb{C}) = \mathcal{H}$ the Dunkl-Watanabe duality, because it bridges the papers [1], [3].

The Watanabe algebra is defined as follows. For $0 \leq i \leq D$, $0 \leq j \leq N - D$, set

$$X_{i,j} := \{x \in \tilde{X} \mid \dim(x) = i + j, \dim(x \cap x_0) = i\}.$$

Then we have a disjoint sum $\tilde{X} = \bigcup X_{i,j}$ over $0 \leq i \leq D$, $0 \leq j \leq N - D$. Let $\tilde{V} := \mathbb{C}\tilde{X}$ denote the vector space over \mathbb{C} with \tilde{X} an orthonormal basis. Then we have a direct sum $\tilde{V} = \bigoplus V_{i,j}$ over the same ranges of i and j , where $V_{i,j}$ is the subspace of \tilde{V} spanned by $X_{i,j}$. The orthogonal projection from \tilde{V} to $V_{i,j}$ is denoted by $E_{i,j}^*$. By identifying the endomorphism algebra $\text{End}(\tilde{V})$ with $M_{\tilde{X}}(\mathbb{C})$, we understand $E_{i,j}^*$ is a diagonal matrix of $M_{\tilde{X}}(\mathbb{C})$.

Define matrices $L_1, L_2, R_1, R_2 \in M_{\tilde{X}}(\mathbb{C})$ by

$$L_1(x, y) := \begin{cases} 1 & \text{if } x \in X_{i-1,j}, y \in X_{i,j}, x \subset y, \\ 0 & \text{otherwise,} \end{cases}$$

$$L_2(x, y) := \begin{cases} 1 & \text{if } x \in X_{i,j-1}, y \in X_{i,j}, x \subset y, \\ 0 & \text{otherwise,} \end{cases}$$

and $R_1 = L_1^t$, $R_2 = L_2^t$, where t denotes the transpose. The Watanabe algebra \mathcal{H} is the subalgebra of $M_{\tilde{X}}(\mathbb{C})$ generated by L_1, L_2, R_1, R_2 together with all $E_{i,j}^*$ ($0 \leq i \leq D, 0 \leq j \leq N - D$).

In [2], irreducible representations of the Terwilliger algebra $T = T(x_0)$ of the q -Johnson scheme $J_q(N, D)$ are determined up to isomorphism using [3]. We plan to use the Dunkl-Watanabe duality as a structural tool to find how far T is from the centralizer algebra for the stabilizer of x_0 in the full automorphism group of $J_q(N, D)$, measured in terms of the coherent length in the framework of the Weisfeiler-Leman stabilization.

References

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