The Turán number of odd prism

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The Turán number of a graph H, $ex(n, H)$, is the maximum number of edges in a graph on n vertices which does not have H as a subgraph. The odd prism $C_{2k+1}\square K_2$ is the cartesian product of C_{2k+1} and K_2. The powerful theorem of Erdős, Stone and Simonovits determines the asymptotic behavior of $ex(n, C_{2k+1}\square K_2)$. In the present paper, we determine the exact value of $ex(n, C_3\square K_2)$. Applying a deep theorem of Simonovits, we characterize the extremal graphs of $ex(n, C_{2k+1}\square K_2)$ for sufficiently large n.

Theorem 1. *The maximum number of edges in an n-vertex $C_3\square K_2$-free graph ($n \neq 5$) is:*

$$ex(n, C_3\square K_2) = \begin{cases} \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor, \ n \equiv 1, 2, 3 \pmod 6, \\ \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil, otherwise. \end{cases}$$

Theorem 2. *Let n be sufficiently large. Then*

$$\mathrm{ex}(n, C_{2k+1}\square K_2) = \max\left\{ n_a(1+n_b) + \frac{1}{2}(j^2 - 3j) : n_a + n_b = n, n_a \equiv j \pmod 3 \right\}.$$

Moreover, all extremal graphs are of the form of a complete bipartite graph with an extremal graph for P_4 added to one of the parts.

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