On Signed Graphs with Fixed Smallest Eigenvalue

Meng-Yue Cao
University of Science and Technology of China, HeFei, P.R. China
caomengyue@ustc.edu.cn

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Let G be a graph with smallest eigenvalue $\lambda_{\min}(G)$. In 1973, Hoffman [?] showed that: (i) for any real number $\lambda \leq -1$, if $\lambda_{\min}(G) \geq \lambda$, then there exists a positive integer $t = t_\lambda$, such that G is $\{K_{1,t}, \widetilde{K_{2t}}\}$-free; (ii) for any integer t, if G is $\{K_{1,t}, \widetilde{K_{2t}}\}$-free, then there exists a positive integer $\lambda = \lambda_t$, such that $\lambda_{\min}(G) \geq \lambda$. In 2016, Kim, Koolen and Yang [?] gave a structure theory for graphs with fixed smallest eigenvalue.

In this talk, I will present a generalization of these results to signed graphs and conclude by discussing an application to signed graphs with smallest eigenvalue greater than $-1-\sqrt{2}$ [?].

Our results are as follows:

Theorem 1. *Let (G,σ) be a signed graph with smallest eigenvalue $\lambda_{\min}(G,\sigma)$. The following hold.*

1. *For any real number $\lambda \leq -1$, there exists a positive integer $t = t(\lambda)$, such that if $\lambda_{\min}(G,\sigma) \geq \lambda$, then (G,σ) is $\{\widetilde{K}_{2t}^{(0)}, \widetilde{K}_{2t}^{(-)}$, $(K_{t+1}, -)$, $(K_{1,t}, +)\}$-switching-free.*
2. *For any positive integer t, there exists a non-positive real number $\lambda = \lambda(t)$, such that if (G,σ) is $\{\widetilde{K}_{2t}^{(0)}, \widetilde{K}_{2t}^{(-)}$, $(K_{t+1}, -)$, $(K_{1,t}, +)\}$-switching-free, then $\lambda_{\min}(G,\sigma) \geq \lambda$.*

Theorem 2. *Let $\lambda \leq -1$ be a real number. There exists a positive integer d_λ such that if (G,σ) is a signed graph with smallest eigenvalue at least λ and minimum valency at least d_λ, then there exists a set of induced subgraphs $N_1, N_2, \ldots, N_r$ of (G,σ), where r is a positive integer, satisfying the following conditions.*

1. *Each vertex of (G,σ) lies in at least one and at most $\lfloor -\lambda \rfloor$ N_i's.*
2. *The induced subgraph N_i is switching equivalent to a signed graph whose positive graph is a $(\lfloor \lambda^2 + 2\lambda + 2 \rfloor)$-plex, for $i = 1, 2, \ldots, r$.*
3. *The intersection $V(N_i) \cap V(N_j)$ contains at most $4\lfloor -\lambda \rfloor - 4$ vertices for $1 \leq i < j \leq r$.*
4. *the subgraph (G', σ') has maximum valency at most $d_\lambda - 1$, where $G' = (V(G), E(G) \setminus \bigcup_{i=1}^{r} E(N_i))$ and $\sigma' = \sigma|_{E(G')}$.*

Let s be a positive integer. A signed graph (G,σ) with smallest eigenvalue $\lambda_{\min}$ is *s-integrable*, if there exists an integer-valued matrix N such that $s(A + \lceil -\lambda_{\min} \rceil \mathbf{I}) = N^T N$, where A is the adjacency matrix of G. Note that (G,σ) is s-integrable if and only if the integral lattice generated by the columns of $\frac{1}{\sqrt{s}}N$ is s-integrable.

Theorem 2. *Let λ be a real number in $(-1-\sqrt{2}, -1]$. There exists a positive integer d'_λ such that if a connected signed graph (G,σ) has smallest eigenvalue $\lambda_{\min}(G,\sigma) \geq \lambda$ and minimum valency at least d'_λ, then $\lambda_{\min}(G,\sigma) \geq -2$ and (G,σ) is 1-integrable.*

References

[1] A.J. Hoffman, On spectrally bounded graphs. *A Survey of Combinatorial Theory* (1973) 277–283.

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[3] J.H. Koolen, J.-Y. Liu, Q. Yang, M.-Y. Cao*, A structure theory for signed graphs with fixed smallest eigenvalue, arXiv:2602.20783.