Geometric structures on knot complements: construction of fundamental polyhedra

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The talk is based on our joint works with Alexander Mednykh and our students

We present a general approach how to construct a canonical fundamental polyhedron for a cone-manifold over some knot in 3-sphere. Then studying the relations of this polyhedron we can establish necessary and sufficient conditions for the existence of such manifolds in both Euclidean and hyperbolic geometries, and derive explicit volume formulas in each case.

Consider cone-manifolds $3_1(\alpha,\gamma)$ whose underlying space is 3-dimensional sphere and whose singular set is the trefoil knot with a bridge with conical angles α and γ along the knot and the bridge respectively. By Wirtinger algorithm we get a presentation of the fundamental group of trefoil knot. Then we find a martix representation of the holonomy group of $3_1(\alpha,\gamma)$ as the group, generated by two rotations by angle α with respect to singular component of the corresponding fundamental set in the Euclidean and in the hyperbolic space. This representation allows us to construct explicitly a canonical fundamental polyhedron for $3_1(\alpha,\gamma)$ in $\mathbb{E}^3$ and $\mathbb{H}^3$. Then we obtain relations between the dihedral angles and the edge lengths of the fundamental polyhedron, that is the relations between the conical angles and the lengths of singular geodesics of $3_1(\alpha,\gamma)$. We establish necessary and sufficient conditions for the existence of $3_1(\alpha,\gamma)$ in $\mathbb{E}^3$ and $\mathbb{H}^3$, and derive explicit volume formulas in each case. Namely, in the hyperbolic case we obtain the following two theorems (see [**?**]).

Theorem 1. *Let $\frac{\pi}{3}<\alpha<\pi$ and let γ satisfy the following equation*

$$\cos^2\frac{\gamma}{2}=\frac{(C^3-A\,B)^2}{C^6},$$

where $A=(4\,\mathcal{X}^2+3\,\mathcal{X}+1)\,\mathcal{Y}-3\,\mathcal{X}^2-\mathcal{X}$, $B=(4\,\mathcal{X}+3)\,\mathcal{Y}^2+(-3\,\mathcal{X}-1)\,\mathcal{Y}+\mathcal{X}$, $C=\mathcal{Y}-\mathcal{X}+2\,\mathcal{X}\,\mathcal{Y}$, $\mathcal{X}=\cosh a$, $\mathcal{Y}=\cos\theta$. *Then the cone-manifold $3_1(\alpha,\gamma)$ admits a hyperbolic structure.*

Moreover, there exists a 12-faced polyhedron in $\mathbb{H}^3$ which is the canonical fundamental set of the cone-manifold $3_1(\alpha,\gamma)$.

Theorem 2. *Let $\mathcal{X}=\cosh a$ and $\mathcal{Y}=\cos\theta$. Then the hyperbolic volume of the cone-manifold $3_1(\alpha,\gamma)$ can be calculated by the formula*

$$Vol(3_1(\alpha,\gamma))=-\frac{1}{2}\int_1^{\mathcal{X}_0}L_\gamma(\mathcal{X})d(\gamma(\mathcal{X})),\quad \textit{where}$$

$$\cosh L_\gamma=\frac{(\mathcal{X}+1)\mathcal{Y}^2+(-\mathcal{X}^2+\mathcal{X}+1)\mathcal{Y}+\mathcal{X}^2-\mathcal{X}-1}{(\mathcal{X}+1)\mathcal{Y}^2+\mathcal{X}^2\mathcal{Y}-\mathcal{X}^2},$$

and $\mathcal{X}_0$ is the only solution of the system of equations

$$\begin{cases}\dfrac{\sinh^2 L_\gamma}{\tan^2\dfrac{\gamma}{4}}\dfrac{\operatorname{csch}^2 L_\alpha}{\tan^2\dfrac{\alpha}{2}}=\dfrac{(\mathcal{X}-\mathcal{Y})^2}{4},\\ \cot^2\dfrac{\alpha}{2}=(2\mathcal{X}+1)(2\mathcal{Y}-1).\end{cases}$$

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References

[1] N. Abrosimov, L. Grunwald, A. Mednykh, A. Qutbaev, B. Vuong, Hyperbolic and Euclidean structures on cone-manifolds over trefoil knot with a bridge. *Sib. Math. J.* **66**:3 (2025) 800–811.