Calderón problem for the fractional Laplacian

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Fractional Laplacian

Definition via Fourier transform:

$$(-\Delta)^s u = \mathscr{F}^{-1}\{|\xi|^{2s}\hat{u}(\xi)\};$$

Definition via Heat equation:

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{U(x,t) - u(x)}{t^{1+s}}\, dt$$

where U solves the heat equation:

$$\partial_t U - \Delta U = 0 \text{ in } \mathbb{R}^n \times (0,\infty),\ \ U|_{t=0} = u \text{ on } \mathbb{R}^n.$$

This operator is *nonlocal*: it does not preserve supports, and computing $(-\Delta)^s u(x)$ involves values of u far away from x.

Fractional Laplacian

Different models for diffusion:

$\partial_t u - \Delta u = 0$	normal diffusion
$\partial_t u + (-\Delta)^s u = 0$	superdiffusion/Lévy flight
$\partial_t^\alpha u - \Delta u = 0$	subdiffusion/CTRW

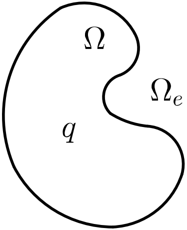
The *fractional Laplacian* is related to

- anomalous diffusion involving long range interactions (turbulent media, population dynamics)
- Lévy processes in probability theory
- financial modelling with jump processes

Fractional Schrödonger equation

Let $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$. Since $(-\Delta)^s$ is nonlocal, the Dirichlet problem becomes

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases} \qquad (*)$$



where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.

Direct problem : Given $f \in H^s(\Omega_e)$, look for a solution $u \in H^s(\mathbb{R}^n)$.

Cauchy data

Neumann data:

$$\mathcal{N}^s u(x)|_{\Omega_e} = c_{n,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy.$$

It satisfies the following limiting relation: [Dipierro, Ros-Oton, Valdinoci 2012]:

$$\lim_{s \to 1} \int_{\Omega_e} \mathcal{N}^s u \, \psi = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \psi, \ \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

Cauchy data:

$$\mathcal{C}_q = \{u|_{\Omega_e}, \mathcal{N}^s u|_{\Omega_e}\}.$$

DN Map: If 0 is not the eigenvalue of (*)

$$\Lambda_q : f \mapsto \mathcal{N}^s u_f.$$

Inverse problem & integral identity

Inverse problem: Given Λ_q, determine q.

Partial data problem: Let
$supp\, u_1 \cap \Omega_e = supp\, u_2 \cap \Omega_e = \widetilde{W} \Subset \Omega_e$ and

$$u_1 = u_2 \text{ in } \widetilde{W}, \quad \Lambda_{q_1} u_1|_W = \Lambda_{q_2} u_2|_W, \; W \Subset \Omega_e.$$

Integral identity:

$$\int_\Omega (q_1 - q_2) u_1\, u_2 = 0.$$

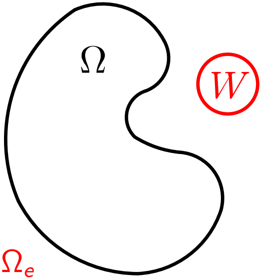
Main result

Theorem (G-Salo-Uhlmann)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W, \widetilde{W} \subset \Omega_e$ are any non-empty open sets, and if

$$\Lambda_{q_1} f|_W = \Lambda_{q_2} f|_W, \quad f \in C_c^\infty(\widetilde{W}),$$

then $q_1 = q_2$ in Ω.



Main features:

- local data result for *arbitrary* $W \subset \Omega_e$
- the same method works for *all* $n \geq 1$
- new mechanism for solving (nonlocal) inverse problems

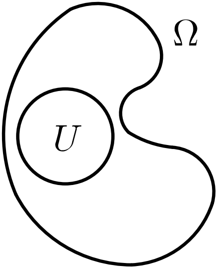
Runge approximation

Classical Runge property (for $\overline{\partial} u = 0$):

analytic functions in simply connected $U \subset \mathbb{C}$ can be approximated by complex polynomials.

General Runge property (for elliptic PDE):

any solution in U, where $U \subset \Omega \subset \mathbb{R}^n$, can be approximated using solutions in Ω.



Main tool: Reduces by duality to the *unique continuation principle* [Lax, Malgrange 1956].

Runge approximation: fractional case

Theorem (G-Salo-Uhlmann)
Any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by *solutions* $u|_\Omega$, where

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad \mathrm{supp}(u) \subset \overline{\Omega} \cup \overline{W}. \qquad (*)$$

Higher order approximation:

If everything is C^∞, any $f \in C^k(\overline{\Omega})$ can be approximated in $C^k(\overline{\Omega})$ by functions $d(x)^{-s} u|_\Omega$ with u as in $(*)$.

(approximate control in all of Ω, this could never hold for Δ).

Main tool: uniqueness

The fractional equation has strong uniqueness properties:

Theorem

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if both u and $(-\Delta)^s u$ vanish in some open set, then $u \equiv 0$.

Such a result could never hold for the Laplacian:
if $u \in C_c^\infty(\mathbb{R}^n)$, then both u and Δu vanish in a large set.

Main tool: uniqueness property

Proof (sketch). If u is nice enough, then

$$(-\Delta)^s u \sim \lim_{y\to 0} y^{1-2s}\partial_y w(\,\cdot\,, y)$$

where $w(x,y)$ is the *Caffarelli-Silvestre extension* of u:

$$\begin{cases} \mathrm{div}_{x,y}(y^{1-2s}\nabla_{x,y} w) = 0 & \text{in } \mathbb{R}^n \times \{y>0\}, \\ w|_{y=0} = u. \end{cases}$$

Thus $(-\Delta)^s u$ is obtained from a *local equation*, which is degenerate elliptic with A_2 weight y^{1-2s}. Carleman estimates [Rüland 2015] and $u|_W = (-\Delta)^s u|_W = 0$ imply uniqueness. □

Single measurement

Theorem (G-Rüland-Salo-Uhlmann)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\frac{1}{4} < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. Let also $W, \widetilde{W} \subset \Omega_e$ be non-empty open sets. If the DN maps for the equations $((-\Delta)^s + q_j)u = 0$ in Ω satisfy

$$\Lambda_{q_1} f|_W = \Lambda_{q_2} f|_W \text{ for a single } f \in C_c^\infty(\widetilde{W}),$$

then $q_1 = q_2$ in Ω.

For, $s \in (0, \frac{1}{4}]$, we recover only continuous bounded potential $q \in C(\Omega) \cap L^\infty$.

Reconstruction from a single-measurement

Theorem (G-Rüland-Salo-Uhlmann)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded open set, let $0 < s < 1$, and let W be an open set with $\overline{\Omega} \cap \overline{W} = \emptyset$. Any function $v \in H^s(\mathbb{R}^n)$ with $\mathrm{supp}(v) \subset \overline{\Omega}$ is uniquely determined by the knowledge of $(-\Delta)^s v|_W$.

Reconstruction algorithm: By writing $u = f + v$ where $v \in \widetilde{H}^s(\Omega)$, and then find $v|_\Omega$ from $(-\Delta)^s v|_W$. So we know $u|_\Omega$ and determine q a.e. in Ω as

$$q := -\frac{(-\Delta)^s u}{u}\Big|_\Omega.$$

Here we use that u can only vanish in a set of measure zero in Ω, and the fact that $f \not\equiv 0$.

Anisotropic fractional Calderón problem

Let $A(x)$ is a C^∞-smooth, elliptic, bounded $n \times n$ matrix defined in $\mathbb{R}^n$. We consider the second order elliptic operator $(-div(A(x)\nabla)$ in $\mathbb{R}^n$ and its fractional power.

- Model problem:

$$\begin{cases} ((-div(A(x)\nabla))^s + q)\, u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e, \end{cases}$$

Theorem (G-Lin-Xiao)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in C(\overline{\Omega})$. Let also $W_1, W_2 \subset \Omega_e$ be non-empty open sets. If the DN maps for the equations $((-div(A(x)\nabla))^s + q_j)\, u_j = 0$ in Ω satisfy

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2} \text{ for a single } f \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω.

Thank you for your attention!