Direct and Inverse Problems for Nonlinear Time-harmonic Maxwell's Equations

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IAS Workshop on Inverse Problems, Imaging and PDE

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Outline

Calderón's Problem

Electrical Impedance Tomography: Recover electric conductivity of an object from voltage-to-current measurements on the boundary.

Posed by Alberto Calderón (1980).

- Voltage-to-current measurements are modeled by the Dirichlet-to-Neumann-map

$$\Lambda_\sigma : f \mapsto \sigma \partial_\nu u|_{\partial\Omega}$$

  where u solves $\nabla \cdot (\sigma \nabla u) = 0$ in Ω, $u|_{\partial\Omega} = f$.

- Inverse Problem: Determine σ from Λ_σ.

Inverse Problem for Maxwell's equations

Consider the time-harmonic Maxwell's equations with a fixed (non-resonance) frequency $\omega > 0$

$$\nabla \times E = i\omega\mu H \quad \text{and} \quad \nabla \times H = -i\omega\varepsilon E \qquad \text{in} \quad \Omega \subset \mathbb{R}^3.$$

- $E, H : \Omega \to \mathbb{C}^3$ electric and magnetic fields;
- $\varepsilon, \mu \in L^\infty(\Omega; \mathbb{C})$ electromagnetic parameters with $\mathrm{Re}(\varepsilon) \geq \varepsilon_0 > 0$ and $Re(\mu) \geq \mu_0 > 0$;
- Electromagnetic measurements on $\partial\Omega$ are modeled by the admittance map
$$\Lambda_{\varepsilon,\mu} : \nu \times E|_{\partial\Omega} \mapsto \nu \times H|_{\partial\Omega}.$$
- Inverse Problem: Determine ε and μ from $\Lambda_{\varepsilon,\mu}$.

Uniquness Results

Conductivity equation:

- Calderón (1980) for the linearized inverse problem;
- Kohn-Vogelius (1985) for piecewise real-analytic conductivities;
- Sylvester-Uhlmann (1987) for smooth conductivities ($n \geq 3$);
- Nachman (1996) for $n = 2$;

Maxwell's equations:

- Somersalo-Isaacson-Cheney (1992) for the linearized inverse problem;
- Ola-Päivärinta-Somersalo (1993);
- Ola-Somersalo (1996) simplified proof;

Nonlinear Conductivity Equations

Consider nonlinear conductivity equation

$$\operatorname{div}(\sigma(x,u,\nabla u)\nabla u) = 0 \qquad \text{in} \quad \Omega \subset \mathbb{R}^n.$$

- $\sigma(x,z,\vec{p}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is positive nonlinear conductivity;
- Measurements on $\partial\Omega$ are given by the nonlinear DN map

  $$\Lambda_\sigma : f \mapsto \sigma(x,u,\nabla u)\partial_\nu u|_{\partial\Omega},$$

  where u solves the above equation with $u|_{\partial\Omega} = f$.
- Inverse Problem: Recover σ from Λ_σ.

Linearization Approach

Due to [Sun] (1996) following [Isakov-Sylvester] (1994). If $\sigma = \sigma(x, u)$, then for a fixed $\lambda \in \mathbb{R}$.

$$\lim_{t \to 0} t^{-1}(\Lambda_\sigma(\lambda + tf) - \Lambda_\sigma(\lambda)) = \Lambda_{\sigma^\lambda}(f), \qquad \sigma^\lambda(x) := \sigma(x, \lambda)$$

in an appropriate norm.

Then the uniqueness problem for the nonlinear equation is reduced to the uniqueness in the linear case:

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \;\Rightarrow\; \Lambda_{\sigma_1^\lambda} = \Lambda_{\sigma_2^\lambda} \quad \text{for all} \;\; \lambda \in \mathbb{R} \;\Rightarrow\; \sigma_1 = \sigma_2.$$

Other Uniqueness Results for Nonlinear Conductivity

For certain $\sigma = \sigma(x, \nabla u)$:

- [Hervas-Sun] (2002) for constant coefficient nonlinear terms and $n = 2$;
- [Kang-Nakamura] (2002) for

$$\sigma(x, \nabla u)\nabla u \quad \text{replaced by} \quad \gamma(x)\nabla u + \sum_{i,j=1}^{n} c_{ij}(x)\partial_i u \partial_j u + R(x, \nabla u).$$

(∗ *Higher Order Linearization.*)

For p-Laplacian type equations: $\sigma = \gamma(x)|\nabla u|^{p-2}$ with $1 < p < \infty$.
(∗ *Linearization is not helpful.*)

- [Salo-Guo-Kar] (2016)
  - under monotonicity condition if $n = 2$;
  - under monotonicity condition for γ close to constant if $n \geq 3$.

Other Nonlinear Equations

Inverse Problems were considered for other nonlinear models:

- Semilinear parabolic: Isakov (1993);
- Semilinear elliptic: Isakov-Sylvester (1994), Isakov-Nachman (1995);
- Elasticity: Sun-Nakamura (1994) for St. Venant-Kirchhoff model;
- Hyperbolic: Lorenzi-Paparoni (1990), Denisov (2007), Nakamura-Vashisth (2017).

** Comparing to the inverse problem of determining spacetime using nonlinear wave interactions.

Outline

Nonlinear Optics

No charge and current density:

$$\nabla \times \mathcal{E} = -\partial_t \mathcal{B}, \quad \nabla \times \mathcal{H} = \partial_t \mathcal{D}, \quad \operatorname{div} \mathcal{D} = 0, \quad \operatorname{div} \mathcal{B} = 0 \quad \text{in } \Omega \times \mathbb{R}.$$

- $\mathcal{E}(t,x)$ and $\mathcal{H}(t,x)$ are electric and magnetic fields;
- $\mathcal{D}$ is the electric displacement and $\mathcal{B}$ is the magnetic induction:

$$\mathcal{D} = \varepsilon \mathcal{E} + \mathcal{P}_{NL}(\mathcal{E}), \quad \mathcal{B} = \mu \mathcal{H} + \mathcal{M}_{NL}(\mathcal{H});$$

  ($*$ *High energy lasers can modify the optical properties of the medium*).
- $\varepsilon, \mu \in L^\infty(\Omega; \mathbb{C})$ are scalar electromagnetic parameters with

$$\mathrm{Re}(\varepsilon) \geq \varepsilon_0 > 0 \quad \text{and} \quad \mathrm{Re}(\mu) \geq \mu_0 > 0;$$

- $\mathcal{P}_{NL}$ and $\mathcal{M}_{NL}$ nonlinear polarization and magnetization.

Nonlinearity

- In nonlinear optics, the polarization $\mathcal{P}(t) = \chi^{(1)}\mathcal{E} + \mathcal{P}_{NL}(\mathcal{E})$

$$\mathcal{P}_{NL} = \chi^{(2)}\mathcal{E}^2 + \chi^{(3)}\mathcal{E}^3 + \cdots := \mathcal{P}^{(2)} + \mathcal{P}^{(3)} + \cdots$$

$*$ $\chi^{(j)}$ — j-th order nonlinear susceptibility.

  - Second-order polarization (Noncentrosymmetric media): incident wave $\mathcal{E} = Ee^{-i\omega t} + c.c.$ generates

$$\mathcal{P}^{(2)}(t) = 2\chi^{(2)}E\overline{E} + \chi^{(2)}E^2e^{-i2\omega t} + c.c.$$

Second harmonic generation

  - Third-order polarization: incident wave $\mathcal{E} = E_1e^{-i\omega_1 t} + E_2e^{-i\omega_2 t} + E_3e^{-i\omega_3 t} + c.c.$ generates polarization with terms of frequencies

$$3\omega_1, 3\omega_2, 3\omega_3, \pm\omega_1 \pm \omega_2 \pm \omega_3, 2\omega_1 + \omega_2, \ldots.$$

  - Sum- and difference-frequency generation.

Nonlinearity

- Lossy media: complex valued.
- Equivalence to time-domain ME:

$$\mathcal{P}^{(2)} = \int_0^\infty \int_0^\infty R^{(2)}(\tau_1, \tau_2) E(t-\tau_1) E(t-\tau_2)\, d\tau_1\, d\tau_2.$$

Using Fourier transform,

$$\chi^{(2)}(\omega_1, \omega_2; \omega_1+\omega_2) = \int_0^\infty \int_0^\infty R^{(2)}(\tau_1, \tau_2) e^{i\omega(\tau_1+\tau_2)}\, d\tau_1\, d\tau_2.$$

Kerr-type Nonlinear Media

We are interested in time-harmonic electromagnetic fields with frequency $\omega > 0$:

$$\mathcal{E}(x,t) = E(x)e^{-i\omega t} + \overline{E(x)}e^{i\omega t}, \qquad \mathcal{H}(x,t) = H(x)e^{-i\omega t} + \overline{H(x)}e^{i\omega t}.$$

A model of nonlinear media of Kerr type:

$$\mathcal{P}_{NL}(x,\mathcal{E}(x,t)) = \chi_e\left(x, \frac{1}{T}\int_0^T |\mathcal{E}(x,t)|^2\,dt\right)\mathcal{E}(x,t) = a(x)|E(x)|^2\mathcal{E}(x,t)$$

$$\mathcal{M}_{NL}(x,\mathcal{H}(x,t)) = \chi_m\left(x, \frac{1}{T}\int_0^T |\mathcal{H}(x,t)|^2\,dt\right)\mathcal{H}(x,t) = b(x)|H(x)|^2\mathcal{H}(x,t).$$

- Kerr-type electric polarization: third order susceptibility $\chi_e^{(3)}(\omega,\omega,\omega;\omega) = a(x)$ common in nonlinear optics;
- Kerr-type magnetization: $\chi_m^{(3)}(\omega,\omega,\omega;\omega) = b(x)$ appears in certain metamaterials;

Maxwell's Equations with the Kerr-type Nonlinearity

This leads to the nonlinear time-harmonic Maxwell's equations

$$\begin{cases} \nabla \times E = i\omega\mu H + i\omega b|H|^2 H \\ \nabla \times H = -i\omega\varepsilon E - i\omega a|E|^2 E. \end{cases} \quad \text{in} \quad \Omega \subset \mathbb{R}^3.$$

Electromagnetic measurements on $\partial\Omega$ are modeled by the admittance map

$$\Lambda^{\omega}_{\varepsilon,\mu,a,b} : \nu \times E|_{\partial\Omega} \mapsto \nu \times H|_{\partial\Omega}.$$

Inverse Problem: Determine ε, μ, a, b from $\Lambda^{\omega}_{\varepsilon,\mu,a,b}$.

Well-posedness for the Direct Problem

Let Div be the surface divergence on $\partial\Omega$. For $1 < p < \infty$, define

$$W^{1-1/p,p}_{\mathrm{Div}}(\partial\Omega) := \{f \in TW^{1-1/p,p}(\partial\Omega) : \ \mathrm{Div}(f) \in W^{1-1/p,p}(\partial\Omega)\},$$

$$W^{1,p}_{\mathrm{Div}}(\Omega) := \{u \in W^{1,p}(\Omega) : \ \nu \times u|_{\partial\Omega} \in W^{1-1/p,p}_{\mathrm{Div}}(\partial\Omega)\}.$$

Theorem (Assylbekov-Z. 2017)

Let $3 < p \le 6$. Suppose $\varepsilon, \mu \in C^2(\overline{\Omega})$ and $a, b \in C^1(\overline{\Omega})$. If $\omega > 0$ is non-resonant, there is $\delta > 0$ such that the b. v. p.

$$\begin{cases} \nabla \times E = i\omega\mu H + i\omega b|H|^2 H \\ \nabla \times H = -i\omega\varepsilon E - i\omega a|E|^2 E \\ \nu \times E|_{\partial\Omega} = f \in W^{1-1/p,p}_{Div}(\partial\Omega) \quad \textit{with} \ \ \|f\|_{W^{1-1/p,p}_{Div}(\partial\Omega)} < \delta \end{cases}$$

has a unique solution $(E, H) \in W^{1,p}_{Div}(\Omega) \times W^{1,p}_{Div}(\Omega)$.

* The proof is based on the Sobolev embedding and the contraction mapping argument.

Main Result for the Inverse Problem

Theorem (Assylbekov-Z. 2017)

Let $4 \le p < 6$*. Suppose* $\varepsilon_j, \mu_j \in C^2(\overline{\Omega})$ *and* $a_j, b_j \in C^1(\overline{\Omega})$, $j = 1, 2$*. Fix a non-resonant* $\omega > 0$ *and small enough* $\delta > 0$*. If*

$$\Lambda^{\omega}_{\varepsilon_1,\mu_1,a_1,b_1}(f) = \Lambda^{\omega}_{\varepsilon_2,\mu_2,a_2,b_2}(f)$$

for all $f \in W^{1-1/p,p}_{Div}(\partial\Omega)$ *with* $\|f\|_{W^{1-1/p,p}_{Div}(\partial\Omega)} < \delta$*, then*

$$\varepsilon_1 = \varepsilon_2, \quad \mu_1 = \mu_2, \quad a_1 = a_2, \quad b_1 = b_2.$$

Brief idea of the proof

- Asymptotic expansion of the admittance map for $s << 1$:

$$\Lambda^{\omega}_{\varepsilon,\mu,a,b}(sf) = s\Lambda^{\omega}_{\varepsilon,\mu}(f) + s^3\nu \times H_2|_{\partial\Omega} + \text{l.o.t.}$$

where (E_2, H_2) solves

$$\begin{aligned} \nabla \times E_2 &= i\omega\mu H_2 + i\omega b|H_1|^2 H_1 \\ \nabla \times H_2 &= -i\omega\varepsilon E_2 - i\omega a|E_1|^2 E_1 \end{aligned} \qquad \nu \times E_2|_{\partial\Omega} = 0.$$

with

$$\begin{aligned} \nabla \times E_1 &= i\omega\mu H_1 \\ \nabla \times H_1 &= -i\omega\varepsilon E_1 \end{aligned} \qquad \nu \times E_1|_{\partial\Omega} = f.$$

- First order linearization gives $\Lambda^{\omega}_{\varepsilon,\mu} : f \mapsto \nu \times H_1|_{\partial\Omega}$ and

$$\Lambda^{\omega}_{\varepsilon_1,\mu_1} = \Lambda^{\omega}_{\varepsilon_2,\mu_2} \Rightarrow \varepsilon_1 = \varepsilon_2, \mu_1 = \mu_2.$$

Brief idea of the proof

- Third order linearization gives the map

$$\partial_s^3 \Lambda^{\omega}_{\varepsilon,\mu,a,b} : f = \nu \times E_1|_{\partial\Omega} \mapsto \nu \times H_2|_{\partial\Omega}.$$

- We derive an integral identity from $\partial_s^3 \Lambda^{\omega}_{\varepsilon,\mu,a_1,b_1} = \partial_s^3 \Lambda^{\omega}_{\varepsilon,\mu,a_2,b_2}$

$$\boxed{\int_\Omega (a_1 - a_2)|E_1|^2 E_1 \cdot \widetilde{E}\, dx + \int_\Omega (b_1 - b_2)|H_1|^2 H_1 \cdot \widetilde{H}\, dx = 0}$$

for all (E_1, H_1) and $(\widetilde{E}, \widetilde{H})$ solving

$$\begin{aligned} \nabla \times E_1 &= i\omega\mu H_1, \\ \nabla \times H_1 &= -i\omega\varepsilon E_1, \end{aligned} \quad \text{and} \quad \begin{aligned} \nabla \times \widetilde{E} &= i\omega\mu \widetilde{H}, \\ \nabla \times \widetilde{H} &= -i\omega\varepsilon \widetilde{E}, \end{aligned}$$

where $\varepsilon = \varepsilon_1 = \varepsilon_2$ and $\mu = \mu_1 = \mu_2$.

Construction of CGO solutions [Ola-Somersalo]

$$\begin{aligned} \nabla \times E - i\omega\mu H &= 0 \\ \nabla \cdot (\mu H) &= 0 \\ \nabla \times H + i\omega\varepsilon E &= 0 \\ \nabla \cdot (\varepsilon E) &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \Phi &= \frac{i}{\omega\mu\varepsilon}\nabla \cdot (\varepsilon E) \\ \nabla \times E - i\omega\mu H - \frac{1}{\varepsilon}\nabla\varepsilon\Psi &= 0 \\ \Psi &= \frac{i}{\omega\mu\varepsilon}\nabla \cdot (\mu H) \\ \nabla \times H + i\omega\varepsilon E + \frac{1}{\mu}\nabla\mu\Phi &= 0 \end{aligned}$$

Set $X = (\mu^{1/2}\Phi, \mu^{1/2}\mathbf{H}^t, \varepsilon^{1/2}\Psi, \varepsilon^{1/2}\mathbf{E}^t)^t$ $\downarrow$ Liouville type of

$\downarrow$ rescaling

$$(P - k + W)X = 0 \quad \text{an elliptic first order system}$$

Construction of CGO solutions

where

$$P = P(D) = \left(\begin{array}{cc|cc} & & & D\cdot \\ & & D & D\times \\ \hline & D\cdot & & \\ D & -D\times & & \end{array}\right)_{8\times 8}, \qquad D = -i\nabla, \quad k = \omega\sqrt{\mu_0\varepsilon_0}.$$

Moreover,

$$X = (P+k-W^t)Y \quad \Rightarrow \quad \boxed{(P-k+W)(P+k-W^t)Y = (-\Delta - k^2 + Q)Y = 0}$$

• *Q is a potential matrix function whose components consist of up to the second order derivatives of μ and ε.*

$$\boxed{\begin{array}{l}(P-k+W)X = 0 \\ X := (x^{(1)}, X^{(2)}, x^{(3)}, X^{(4)})\end{array}} \quad \xRightarrow{x^{(1)}=x^{(3)}=0} \quad \boxed{\begin{array}{c}(\varepsilon^{-1/2}X^{(4)}, \mu^{-1/2}X^{(2)}) \\ \text{is the solution to Maxwell.}\end{array}}$$

CGO solutions Y_{CGO} to the Schrödinger eqution

$$Y_{\mathrm{CGO}}(x) = e^{\tau(\varphi(x)+i\psi(x))}(A(x)+R(x)), \quad R = o_{\tau\to\infty}(1)A.$$

Main Steps:

- Choose φ to be a Limiting Carleman Weight (LCW):

$$\langle \varphi'' \nabla\varphi, \nabla\varphi\rangle + \langle \varphi''\xi, \xi\rangle = 0$$

  when $|\xi|^2 = |\nabla\varphi|^2$ and $\xi \cdot \nabla\varphi = 0$.
- Eikonal equation for ψ: $|\nabla\psi|^2 = |\nabla\varphi|^2$, $\nabla\psi\cdot\nabla\varphi = 0$.
- Then

$$\begin{aligned}(\mathcal{L}_{\varphi+i\psi} + Q)R =& -(\mathcal{L}_{\varphi+i\psi} + Q)A \\ &= (\Delta + k^2 - Q)A + \tau[2\nabla(\varphi+i\psi)\cdot\nabla + \Delta(\varphi+i\psi)]A\end{aligned}$$

  where $\mathcal{L}_{\varphi+i\psi} := e^{-\tau(\varphi+i\psi)}(-\Delta - k^2)e^{\tau(\varphi+i\psi)}$.

  Choose A solving $[2\nabla(\varphi+i\psi)\cdot\nabla + \Delta(\varphi+i\psi)]A = 0$.

CGO solutions to the Schrödinger equation

- Last step: Solving

$$\boxed{(\mathcal{L}_{\varphi+i\psi} + Q)R = (\Delta + k^2 - Q)A}$$

- [Sylvester-Uhlmann,1987]: $\tau(\varphi + i\psi) = \zeta \cdot x$ where $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = -k^2$, $|\zeta| \sim \tau$. A is constant. Solve

$$(-\Delta - 2\zeta \cdot \nabla + Q)R = -QA$$

globally using estimate $\|(-\Delta - 2\zeta \cdot \nabla)^{-1}\| \sim O(\tau^{-1})$.
- [Kenig-Sjöstrand-Uhlmann,2007]: Nonlinear LCW $\varphi(x) = -\ln|x|$
- [Kenig-Salo-Uhlmann,2012]: LCW $\varphi(x) = -x_1$, $\psi(x)$ is linear in the transverse polar coordinate $r = (x_2^2 + x_3^2)^{1/2}$.

$\varphi(x) = -x_1$ and $\varphi(x) = -\ln|x|$

Set $z = x_1 + ir$, where $x = (x_1, r, \theta)$ denotes the cylindrical coordinate.

- LCW $\varphi(x) = -x_1 = \Re(z) \quad \to \quad \varphi + i\psi = l(z) = -z.$

$$l'(z)\left(2\partial_{\bar z} - \frac{1}{z-\bar z}\right)A = 0, \qquad \text{we take } A = \frac{e^{i\lambda z}}{\sqrt{2ir}}g(\theta).$$

- LCW $\varphi(x) = -\ln|x| = -\Re(\ln\bar z) \quad \to \quad \varphi + i\psi = l(\bar z) = -\ln\bar z.$

$$l'(\bar z)\left(2\partial_{z} + \frac{1}{z-\bar z}\right)A = 0, \qquad \text{we take } A = \frac{e^{i\lambda \bar z}}{\sqrt{2ir}}g(\theta).$$

► In both cases, a semi-classical Carleman estimate on a bounded domain can be derived as an a-priori estimate for the adjoint operator $\mathcal{L}^*_{\varphi+i\psi}$ (see e.g. [Kenig-Sjöstrand -Uhlmann,2007]), in order to prove both the **existence** and **smallness** of R (relative to A) for τ large.

► $g(\theta)$ is an arbitrary 8-vector function.

CGO solutions for Maxwell's equations

We have

$$\begin{aligned} X_{\mathrm{CGO}} &= (P + k - W^t)[e^{\tau(\varphi+i\psi)}(A+R)] \\ &= e^{\tau(\varphi+i\psi)}\{\tau P(D(\varphi+i\psi))A + (P+k-W^t)A \\ &\qquad + \tau P(D(\varphi+i\psi))R + (P+k-W^t)R\} \\ &:= e^{\tau(\varphi+i\psi)}(B+S) \end{aligned}$$

Recall that we need $x^{(1)} = x^{(3)} = 0$.

- Choose $g(\theta)$ such that $b^{(1)} = b^{(3)} = 0$.
- To show $s^{(1)} = s^{(3)} = 0$, use

$$(P+k-W^t)(P-k+W)X = (-\Delta - k^2 + \widetilde{Q})X = 0$$

where only $\widetilde{Q}_{11}$ and $\widetilde{Q}_{33}$ are nonzero in their corresponding rows.

$$\Longrightarrow \quad (\mathcal{L}_{\varphi+i\psi} + \widetilde{Q}_{kk})s^{(k)} = 0, \qquad k = 1, 3.$$

$$\overset{?}{\Longrightarrow} s^{(k)} = 0.$$

Uniqueness for $\mathcal{L}_{\varphi+i\psi}$

- For linear phase $\zeta \cdot x$, uniqueness is obtained globally assuming decaying at infinity. (see [Sylvester-Uhlmann, 1987]);
- For $\varphi(x) + i\psi(x) = -(x_1 + ir)$, uniqueness is obtained on a cylinder $T = \mathbb{R} \times B(0, R)$ for $B(0, R) \subset \mathbb{R}^2$ assuming zero Dirichlet condition and decaying at infinities of x_1 direction. (see [Kenig-Salo-Uhlmann, 2012]);
- For $\varphi(x) + i\psi(x) = -\ln \bar{z}$, following [Nachman-Street, 2002], we decompose $L^2(\Omega)$ as

$$\overline{\mathcal{L}^*_{\varphi+i\psi}\{v \in \mathcal{S}(\mathbb{R}^n) : \text{supp}(v) \subset \Omega.\}} \oplus W$$

Then we can show the uniqueness by choosing S as above.

Try CGO solutions with linear phases

- Recall the integral identity

$$\int_\Omega (a_1 - a_2)|E_1|^2 E_1 \cdot \widetilde{E}\, dx + \int_\Omega (b_1 - b_2)|H_1|^2 H_1 \cdot \widetilde{H}\, dx = 0$$

- Complex Geometrical Optics solutions with linear phases

$$\begin{aligned} E_1(x) &= \varepsilon^{-1/2} e^{\zeta_1 \cdot x}\left(s_0 \frac{\zeta_1 \cdot \alpha}{|\zeta_1|}\zeta_1 + o_{|\zeta_1|\to\infty}(|\zeta_1|)\right) \\ H_1(x) &= \mu^{-1/2} e^{\zeta_1 \cdot x}\left(t_0 \frac{\zeta_1 \cdot \beta}{|\zeta_1|}\zeta_1 + o_{|\zeta_1|\to\infty}(|\zeta_1|)\right) \end{aligned} \quad \text{as } |\zeta_1| \to \infty.$$

where for $\xi = (\xi_1, 0, 0)$, $\xi_1 \in \mathbb{R}$,

$$\zeta_1 = \left(i\xi_1, -\sqrt{\xi_1^2/4 + \tau^2}, i\sqrt{\tau^2 - k^2}\right) \Rightarrow \zeta_1 \cdot \zeta_1 = k^2, |\zeta_1| \sim \tau.$$

  - $s_0 = 1, t_0 = 0$: the leading term of $|E_1|^2 E_1 \cdot \widetilde{E}$ does not provide enough F.T. of $(a_1 - a_2)$;

Polarization

In the integral identity, we plug in

$$(E_1, H_1) = (E_{(1)} + E_{(2)} + E_{(3)}, H_{(1)} + H_{(2)} + H_{(3)})$$

where $(E_{(j)}, H_{(j)})$ are solutions to the linear equations. Then

$$\begin{aligned} &\int_\Omega (a_1 - a_2)\Big[\Re(E_{(3)} \cdot \overline{E}_{(2)})(E_{(1)} \cdot \overline{\widetilde{E}}) + \Re(E_{(3)} \cdot \overline{E}_{(1)})(E_{(2)} \cdot \overline{\widetilde{E}}) \\ &+ \Re(E_{(1)} \cdot \overline{E}_{(2)})(E_{(3)} \cdot \overline{\widetilde{E}})\Big]\, dx \\ &+ \int_\Omega (b_1 - b_2)\Big[\Re(H_{(3)} \cdot \overline{H}_{(2)})(H_{(1)} \cdot \overline{\widetilde{H}}) + \Re(H_{(3)} \cdot \overline{H}_{(1)})(H_{(2)} \cdot \overline{\widetilde{H}}) \\ &+ \Re(H_{(1)} \cdot \overline{H}_{(2)})(H_{(3)} \cdot \overline{\widetilde{H}})\Big]\, dx = 0 \end{aligned}$$

Continue the proof

- $s_0 = 1, t_0 = 0$ and

$$E_{(1)} = e^{-\tau(x_1+ir)}\left[s_0\,\varepsilon^{-1/2}e^{i\lambda(x_1+ir)}\eta(\theta)(dx_1+idr)+R_{(1)}\right],$$
$$E_{(2)} = e^{\tau(x_1-ir)}\left[\varepsilon^{-1/2}e^{i\lambda(x_1-ir)}(dx_1-idr)+R_{(2)}\right],$$
$$E_{(3)} = e^{-\tau(x_1-ir)}\left[\varepsilon^{-1/2}e^{-i\lambda(x_1-ir)}(dx_1-idr)+R_{(3)}\right],$$
$$\widetilde{E} = e^{\tau(x_1+ir)}\left[\varepsilon^{-1/2}e^{i\lambda(x_1+ir)}(dx_1+idr)+\widetilde{R}\right].$$

- Decay of the remainders ($4 \le p < 6$)

$$\|R_{(j)}\|_{L^p(\Omega)}, \|R\|_{L^p(\Omega)} \le C\frac{1}{|\tau|^{\frac{6-p}{2p}}}, \quad j = 1,2,3.$$

Continue the proof

- Plugging in the integral identity, as $\tau \to \infty$,

$$\int \left(\frac{(a_1 - a_2)\chi_\Omega}{|\varepsilon|^2} \right) e^{-i2\lambda(x_1 - ir)} \eta(\theta)\, dx_1\, dr\, d\theta = 0.$$

- Set $f := \frac{(a_1 - a_2)\chi_\Omega}{|\varepsilon|^2}$ and let $\eta(\theta) \in C^\infty(\mathbb{S}^1)$ vary.

$$\int_0^\infty e^{-2\lambda r} \hat{f}(2\lambda, r, \theta)\, dr = 0, \quad \theta \in \mathbb{S}^1.$$

Attenuated geodesic ray transform on $\mathbb{R}^2$

$$\Rightarrow \widehat{f} = 0 \qquad\qquad \Rightarrow a_1 = a_2.$$

- Generalization to the inverse problems on admissible transversally anisotropic manifolds.

Extended result

In [Cârstea, 2018], the framework is extended to prove the uniqueness in determining

$$\mathcal{P}_{NL} = \sum_{k=1}^{\infty} a_k(x)|E|^{2k}\mathcal{E}, \quad \mathcal{M}_{NL} = \sum_{k=1}^{\infty} b_k(x)|H|^{2k}\mathcal{H}.$$

$$\boxed{\begin{aligned}&\int_\Omega (a_k - a_k')\left[(e_0\cdot e_1)(e_2\cdot \overline{e}_3)^k + k(e_0\cdot e_2)(e_1\cdot \overline{e}_3)(e_2\cdot \overline{e}_3)^{k-1}\right]\\ &-(b_k - b_k')\left[(h_0\cdot h_1)(h_2\cdot \overline{h}_3)^k + k(h_0\cdot h_2)(h_1\cdot \overline{h}_3)(h_2\cdot \overline{h}_3)^{k-1}\right]\ dx = 0.\end{aligned}}$$

Nonlinearity in Second Harmonic Generation

- Incident beam: $E(t,x) = E_0(x)e^{-it\omega} + c.c.$
- Writing the solution to include terms at frequency ω and 2ω:

$$E(t,x) = 2\mathrm{Re}\left\{E^{\omega}(x)e^{-i\omega t}\right\} + 2\mathrm{Re}\left\{E^{2\omega}(x)e^{-i2\omega t}\right\}$$
$$H(t,x) = 2\mathrm{Re}\left\{H^{\omega}(x)e^{-i\omega t}\right\} + 2\mathrm{Re}\left\{H^{2\omega}(x)e^{-i2\omega t}\right\},$$

- Then we obtain the system

$$\begin{cases} \nabla \times E^{\omega} - i\omega\mu H^{\omega} = 0, \\ \nabla \times H^{\omega} + i\omega\varepsilon E^{\omega} + i\omega\chi^{(2)}\overline{E^{\omega}} \cdot E^{2\omega} = 0, \\ \nabla \times E^{2\omega} - i2\omega\mu H^{2\omega} = 0, \\ \nabla \times H^{2\omega} + i2\omega\varepsilon E^{2\omega} + i2\omega\chi^{(2)} E^{\omega} \cdot E^{\omega} = 0. \end{cases} \tag{1}$$

∗ *One can also introduce the similar nonlinear second harmonic generation effect for magnetic fields;*
∗ *Here we can also assume that μ, ε depend on the frequency.*

Applications

- SHG can be so efficient that nearly all of the power in the incident beam at frequency ω is converted to radiation at the frequency 2ω;
- Second harmonic generation microscopy for noncentrosymmetric media.

Well-posedness for the Direct Problem

Theorem

Let $3 < p \leq 6$. Suppose that $\varepsilon, \mu \in C^2(\Omega;\mathbb{C})$ are complex-valued functions with positive real parts and $\chi^{(2)} \in C^1(\Omega;\mathbb{R}^3)$. For every $\omega \in \mathbb{C}$, outside a discrete set $\Sigma \subset \mathbb{C}$ of resonant frequencies, there is $\delta > 0$ such that for a pair $(f^\omega, f^{2\omega}) \in \left(TW^{1-1/p,p}_{Div}(\partial M)\right)^2$ with $\sum_{k=1,2} \|f^{k\omega}\|_{TW^{1-1/p,p}_{Div}(\partial\Omega)} < \delta$, the Maxwell's equations (1) has a unique solution $(E^\omega, H^\omega, E^{2\omega}, H^{2\omega}) \in \left(W^{1,p}_{Div}(\Omega)\right)^4$ satisfying $\nu \times E^{k\omega}|_{\partial\Omega} = f^{k\omega}$ for $k = 1,2$ and

$$\sum_{k=1,2} \|E^{k\omega}\|_{W^{1,p}_{Div}(\Omega)} + \|H^{k\omega}\|_{W^{1,p}_{Div}(\Omega)} \leq C \sum_{k=1,2} \|f^{k\omega}\|_{TW^{1-1/p,p}_{Div}(\partial\Omega)},$$

for some constant $C > 0$ independent of $(f^\omega, f^{2\omega})$.

Inverse Problem

- **Admittance Map**:

$$\Lambda^{\omega,2\omega}_{\varepsilon,\mu,\chi^{(2)}}(f^{\omega},f^{2\omega}) = \left(\nu \times H^{\omega}|_{\partial\Omega},\ \nu \times H^{2\omega}|_{\partial\Omega}\right),$$

Theorem

Let $4 \leq p < 6$. Suppose that $\varepsilon_j \in C^3(\Omega;\mathbb{C})$, $\mu_j \in C^2(\Omega;\mathbb{C})$ with positive real parts and $\chi_j^{(2)} \in C^1(\Omega;\mathbb{R}^3)$, $j = 1,2$. Fix $\omega > 0$ outside a discrete set of resonant frequencies $\Sigma \subset \mathbb{C}$ and fix sufficiently small $\delta > 0$. If

$$\Lambda^{\omega,2\omega}_{\varepsilon_1,\mu_1,\chi_1^{(2)}}(f^{\omega},f^{2\omega}) = \Lambda^{\omega,2\omega}_{\varepsilon_2,\mu_2,\chi_2^{(2)}}(f^{\omega},f^{2\omega})$$

for all $(f^{\omega},f^{2\omega}) \in \left(TW^{1-1/p,p}_{Div}(\partial\Omega)\right)^2$ *with* $\sum_{k=1,2} \|f^{k\omega}\|_{TW^{1-1/p,p}_{Div}(\partial\Omega)} < \delta$,

then $\varepsilon_1 = \varepsilon_2$, $\mu_1 = \mu_2$, $\chi_1^{(2)} = \chi_2^{(2)}$ *in* Ω.

Brief idea of the proof

- Asymptotic expansion of the admittance map for $s << 1$:

$$\Lambda^{\omega,2\omega}_{\varepsilon,\mu,\chi^{(2)}}(sf^{\omega}, sf^{2\omega}) = s\Lambda^{\omega,2\omega}_{\varepsilon,\mu}(f^{\omega}, f^{2\omega}) + s^2(\nu\times H_2^{\omega}|_{\partial\Omega}, \nu\times H_2^{2\omega}|_{\partial\Omega}) + \text{l.o.t.}$$

where $(E_2^{\omega}, H_2^{\omega}, E_2^{2\omega}, H_2^{2\omega})$ solves

$$\begin{cases} \nabla\times E_2^{k\omega} - ik\omega\mu H_2^{k\omega} = 0, \quad k = 1, 2, \\ \nabla\times H_2^{\omega} + i\omega\varepsilon E_2^{\omega} + i\omega\chi^{(2)}\overline{E_1^{\omega}}\cdot E_1^{2\omega} = 0, \\ \nabla\times H_2^{2\omega} + i2\omega\varepsilon E_2^{2\omega} + i\omega\chi^{(2)}E_1^{\omega}\cdot E_1^{\omega} = 0, \\ \nu\times E_2^{k\omega}|_{\partial\Omega} = 0, \quad k = 1, 2. \end{cases}$$

with

$$\nabla\times E_1^{k\omega} - ik\omega\mu H_1^{k\omega} = 0, \quad \nabla\times H_1^{k\omega} + ik\omega\varepsilon E_1^{k\omega} = 0,$$
$$\nu\times E_1^{k\omega}|_{\partial\Omega} = f^{k\omega}, \quad k = 1, 2.$$

- Linearization gives $\Lambda^{\omega,2\omega}_{\varepsilon_1,\mu_1} = \Lambda^{\omega,2\omega}_{\varepsilon_2,\mu_2} \Rightarrow \varepsilon_1 = \varepsilon_2$ and $\mu_1 = \mu_2$;

Brief idea of the proof

- Using the second order term, the map

$$(f^{\omega}, f^{2\omega}) \mapsto (\nu \times H_2^{\omega}|_{\partial\Omega}, \nu \times H_2^{2\omega}|_{\partial\Omega})$$

we derive a nonlinear integral identity

$$\boxed{\int \chi_{\Omega}\left(\chi_1^{(2)} - \chi_2^{(2)}\right) \cdot \left[\left(\overline{E_1^{\omega}} \cdot E_1^{2\omega}\right)\widetilde{E}^{\omega} + 2\left(E_1^{\omega} \cdot E_1^{\omega}\right)\widetilde{E}^{2\omega}\right] dx = 0}$$

for all $(E_1^{\omega}, H_1^{\omega}, E_1^{2\omega}, H_1^{2\omega})$ and $(\widetilde{E}^{\omega}, \widetilde{H}^{\omega}, \widetilde{E}^{2\omega}, \widetilde{H}^{2\omega})$ solving the linear equations with $\mu = \mu_1 = \mu_2$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$.

  - *No polarization is needed since linear equations for E^{ω} and $E^{2\omega}$ are decoupled.*

- Plug in CGO solutions from [Ola-Päivärinta-Sommersalo],

Continue the proof

- For $\xi = \xi_1 e_1$, choose

$$\zeta_1^\omega = \begin{pmatrix} i\frac{\xi_1}{2} \\ -\sqrt{\frac{\xi_1^2}{4} + \tau^2} \\ i\sqrt{\tau^2 - k^2} \end{pmatrix}, \quad E_1^\omega = e^{\zeta_1^\omega \cdot x}\left[\begin{pmatrix} 1 \\ 1 \\ \frac{\sqrt{\frac{\xi_1^2}{4}+\tau^2} - i\frac{\xi_1}{2}}{i\sqrt{\tau^2-k^2}} \end{pmatrix} + R_1^\omega\right]$$

$$\zeta_1^{2\omega} = \begin{pmatrix} -i\xi_1 \\ 2\sqrt{\frac{\xi_1^2}{4} + \tau^2} \\ 2i\sqrt{\tau^2 - k^2} \end{pmatrix}, \quad E_1^{2\omega} = e^{\zeta_1^{2\omega} \cdot x}\left[\begin{pmatrix} 1 \\ 1 \\ \frac{i\xi_1 - 2\sqrt{\frac{\xi_1^2}{4}+\tau^2}}{2i\sqrt{\tau^2-k^2}} \end{pmatrix} + R_1^{2\omega}\right]$$

$$\widetilde{\zeta}^\omega = \begin{pmatrix} i\frac{\xi_1}{2} \\ -\sqrt{\frac{\xi_1^2}{4} + \tau^2} \\ -i\sqrt{\tau^2 - k^2} \end{pmatrix}, \quad \widetilde{E}^\omega = e^{\widetilde{\zeta}^\omega \cdot x}\left[\widetilde{A}^\omega + \widetilde{R}^\omega\right]$$

$$\boxed{\left(\overline{E_1^\omega} \cdot E_1^{2\omega}\right)\widetilde{E}^\omega = e^{-i\xi\cdot x}(\widetilde{A}^\omega + l.o.t.)}$$

- Choose $\widetilde{\zeta}^{2\omega}$ such that $2(E_1^\omega \cdot E_1^\omega)\widetilde{E}^{2\omega}$ decays exponentially in τ.

Continue the proof

$$\boxed{\mathcal{F}\left(\chi_\Omega(\chi_2^{(2)} - \chi_1^{(2)}) \cdot \widetilde{A}_\infty^{2\omega}\right)(\xi) = 0}$$

where $\widetilde{A}_\infty^{2\omega} = \lim_{\tau\to\infty} \widetilde{A}^{2\omega}$.
Choose enough $\widetilde{A}^{2\omega}$ so that $\chi_\Omega(\chi_2^{(2)} - \chi_1^{(2)}) = 0$.

(*end of the proof.*)

- The *higher order linearization* approach can be used to reconstruct any order harmonic generation in nonlinear optics.

Some recent scalar case results (semi-linear equations)

- [Lassas-Liimantainen-Lin-Salo], [Feizmohammadi-Oksanen]

$$\Delta u + a(x, u) = 0, \qquad x \in \Omega$$

where $a(x, z)$ satisfies

  - $a(x, 0) = 0$ and 0 is not a Dirichlet eigenvalue to $\Delta + \partial_z a(x, 0)$ in Ω
  - $a(x, 0) = \partial_z a(x, 0) = 0$ (*Using non-CGO solutions, [Krupchyk-Uhlmann]*)
  - $a(x, z)$ is analytic in z: $a(x, z) = \sum_{k=1}^{\infty} a_k(x) \frac{z^k}{k!}$.

($*$ *Simultaneously reconstruct inclusions/obstacles and the background coefficients; Partial data problems*)

Future work

- Implement other solutions.
- Partial data problems.

Thanks for your attention!