Corners Scattering and Inverse Scattering

Jingni Xiao

Department of Mathematics
Rutgers University

Joint with Emilia Blåsten, Fioralba Cakoni and Hongyu Liu

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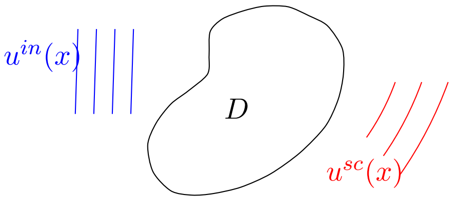
Scattering



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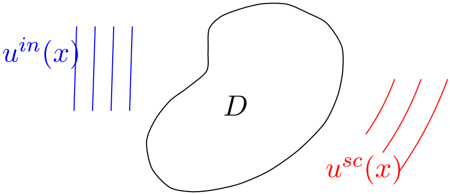

- Inverse scattering
- Invisibility

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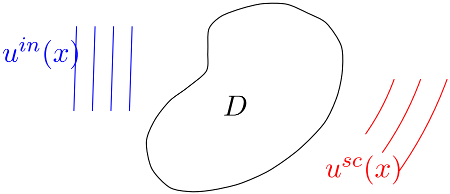
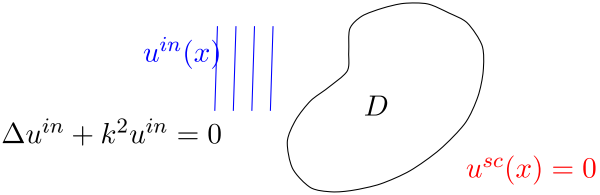


- Inverse scattering
- Invisibility

Question: $\exists\ \{[D; u^{in}]\}$,

s.t. $u^{sc} \equiv 0$ in D^c (or equiv., $u_\infty \equiv 0$) ?

A resulting problem



- Interior Transmission Eigenvalue Problem

$$\begin{aligned} \nabla \cdot a \nabla u + k^2 c u = 0, \quad \Delta v + k^2 v = 0, \qquad & \text{in } D, \\ u = v, \quad a \partial_\nu u = \partial_\nu v, \qquad & \text{on } \partial D. \end{aligned}$$

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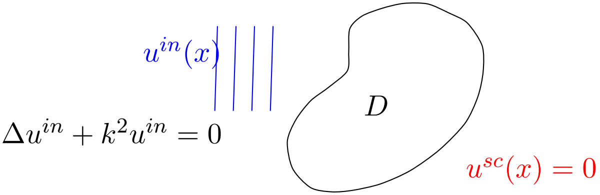

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- Equivalence?

Some known result

Certain shapes: invisible under certain probing waves:

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$\Delta + k^2 q$: [Blåsten-Päivärinta-Sylvester '14,
Päivärinta-Salo-Vesalainen '17, Elschner-Hu '15&'18,
Hu-Salo-Vesalainen '16, Blåsten-Vesalainen '18, etc.]
Source problem: [Blåsten '18]

Maxwell's equations: [Liu-X. '17 (Right corner)]

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Cloaking: invisible for all probing fields

- Anisotropic and singular medium [Greenleaf-Lassas-Uhlmann '03, etc.]

The scattering problem

Consider the scattering problem

$$\nabla \cdot a\nabla u + k^2 cu = 0 \quad \text{in } \mathbb{R}^n,$$

$$\Delta u^{in} + k^2 u^{in} = 0 \quad \text{in } \mathbb{R}^n,$$

$$\hat{x} \cdot \nabla u^{sc} - iku^{sc} = o(|x|^{\frac{n-1}{2}}), \quad |x| \to \infty.$$

where $a, c \in L^\infty(\mathbb{R}^n)$, $a = (a_{ij})$ symm. and positive definite, and $a - I$ and $c - 1$ compactly supported,

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Corner scattering:

- Convex corner(s) at the support of $a - I$ or/and that of $c - 1$;
- Around the corner, a is locally $W^{3,1+\varepsilon}$ and scalar, and/or c is locally $W^{1,1+\varepsilon}$.

Corner Scatters

Theorem 1 ([Cakoni-X '19])

(Roughly) Corners of c with a jump across the corner always scatter any incident wave, when $a-1$ vanishes to the second order at the same corner.

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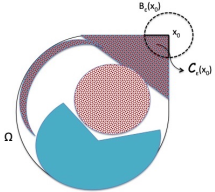

Figure: dotted: supp$(c-1)$; colored: supp$(a-I)$.

$$(\rho(x)-1)\gamma^{-1/2}(x)=\rho_0+O(|x-x_0|^\sigma),\ (\gamma(x)-1)\,\gamma^{-1/2}(x)=O(|x-x_0|^{2+\sigma}),$$

where $\rho_0=const\neq 0$, $(\gamma,\rho)|_{\mathcal{C}_\varepsilon(x_0)}=(a,c)|_{\mathcal{C}_\varepsilon(x_0)}$ in $W^{3,1+\varepsilon}\times W^{1,1+\varepsilon}$.

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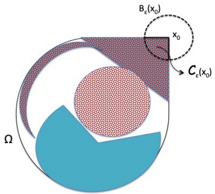

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where $\rho_0 = const \neq 0$, $(\gamma,\rho)|_{\mathcal{C}_\varepsilon(x_0)} = (a,c)|_{\mathcal{C}_\varepsilon(x_0)}$ in $W^{3,1+\varepsilon} \times W^{1,1+\varepsilon}$.
Generalization of the previous results concerning the operator $\Delta + k^2 q$.

Corner Scatters

Other cases:

- Corners of $c-1$ with a jump across the corner ($\rho_0 \neq 0$).
  - If $a-1$ vanishes to the first order at the same corner, i.e., $(\gamma(x)-1)\,\gamma^{-1/2}(x) = O(|x-x_0|^{1+\sigma})$, then the corner scatters all incident fields u^{in} for which $N_T u^{in} = N_T \nabla u^{in}$.

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  - If $a-1$ vanishes at the corner, i.e., $(\gamma(x)-1)\,\gamma^{-1/2}(x) = O(|x-x_0|^{\sigma})$, then the corner scatters all incident fields u^{in} satisfying $u^{in}(x_0) \neq 0$ and $\nabla u^{in}(x_0) = 0$.

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  $(\gamma(x)-1)\,\gamma^{-1/2}(x) = \gamma_0 + O(|x-x_0|^{\sigma})$ with $\gamma_0 = const \neq 0$. If the corner is of aperture $\kappa\pi$, $\kappa \in (0,1)$, then it scatters all incident fields u^{in} for which $N_T \nabla u^{in} \neq \kappa l - 1$ with some $l \in \mathbb{N}_+$.
  e.g.: $\nabla u^{in}(x_0) \neq 0$.

Shape Determination

Recall: forward scattering:

$$\nabla \cdot a\nabla u + k^2 cu = 0 \quad \text{in } \mathbb{R}^n,$$

where $a - I$ and $c - 1$ are compactly supported and $u = u^{in} + u^{sc}$ with u^{in} the incident field and u^{sc} the scattered field.

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The inverse problem: To uniquely determine the convex hull of $c - 1$ or/and $a - 1$ from the far-field measurement u^{∞} or from the near-field (scattered) measurement $u^{sc}|_{\partial B_{2R}}$.

Unique determination from a single measurement

Assumptions (roughly):

1. The convex hull D of $\text{supp}(c-1)$ is a bounded polygon, $\text{supp}(a-I) \subseteq D$.
2. Local regularity of c and a ($W^{1,1+\varepsilon}$ and $W^{3,1+\varepsilon}$) around corners of D.
3. $c-1$ has a jump while $a-1$ vanishes to the second order at each corner.

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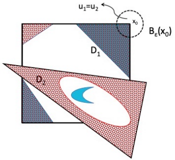
Theorem 2 (Cakoni-X '19)

D can be uniquely determined from the far-field (or scattered) data u^∞ corresponding to a single incident field.



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ITEP and Herglotz functions

Interior transmission eigenvalue problem:

$$\begin{aligned} \nabla \cdot a\nabla u + k^2 cu = 0, \quad \Delta v + k^2 v = 0, \qquad & \text{in } D, \\ u = v, \quad a\partial_\nu u = \partial_\nu v, \qquad & \text{on } \partial D. \end{aligned}$$

Herglotz wave functions:

$$v_g(x) = \int_{\mathbb{S}^{n-1}} g(d) e^{ikx\cdot d} ds_d, \qquad g \in L^2(\mathbb{S}^{n-1}).$$

Fact ([Cakoni-Colton-Haddar '16 (Book), etc.]): the set of Herglotz wave functions is dense in

$$\{v \in H^1(\Omega) : \Delta v + k^2 v = 0 \quad \text{in } \Omega\}.$$

Blow-up of the kernel

To approximate v (eigenfunctions) by v_g.

Question: Do the Herglotz kernels g_ε keep bounded when $v_{g_\varepsilon} \to v$?

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Lemma 1 (Cakoni-X. '19)

When the support of $c-1$ or/and $a-I$ has a corner, and a and c satisfies (roughly) the local conditions at the corner, then

$$\limsup \|g_\varepsilon\|_{L^2(\mathbb{S}^1)} = \infty$$

If non-scattering happens

$$\begin{aligned} \nabla\cdot a\nabla u + k^2 cu = 0, \quad & \Delta u^{in} + k^2 u^{in} = 0, & \text{in } D,\\ u = u^{in}, \quad & a\partial_\nu u = \partial_\nu u^{in}, & \text{on } \partial D. \end{aligned}$$

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A resulting local problem at the corner:

$$\begin{aligned} \nabla \cdot \gamma\nabla u + k^2 \rho u = 0, \quad \Delta v + k^2 v = 0, \qquad & \text{in } \mathcal{C}_\varepsilon, \\ u = v, \quad \gamma\partial_\nu u = \partial_\nu v, \qquad & \text{on } \partial\mathcal{C}_\varepsilon \setminus \mathcal{K}_\varepsilon. \end{aligned}$$

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An integral identity:

$$\int_{\mathcal{C}_\varepsilon} (\gamma-1)\nabla v\cdot\nabla w - k^2(\rho-1)vw\,dx = \int_{\mathcal{K}_\varepsilon} \gamma\partial_\nu w(v-u) - w\left(\partial_\nu v - \gamma\partial_\nu u\right)\,ds$$

for any solution w to

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Later: asymptotic analysis of the integrals.

CGO solutions

$$\nabla \cdot \gamma \nabla w + k^2 \rho w = 0 \quad \text{in } \mathbb{R}^n.$$

Solutions of the form

$$w = w_\tau = \gamma^{-1/2}(1 + r(x))e^{\eta \cdot x}, \tag{4.1}$$

with $\eta = -\tau \left(d + i d^\perp\right)$ and $d, d^\perp \in \mathbb{S}^{n-1}$ satisfying $d \cdot d^\perp = 0$.

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Proposition 1 (Cakoni-X, '19)

(In short) Given $n = 2$, $s \in \{0, 1\}$, $\varepsilon > 0$, *there exist* $p > 3$ *and* $\sigma > 0$ *such that for any* $\gamma \in H^{3,1+\varepsilon}(\mathbb{R}^n)$, $\rho \in H^{1,1+\varepsilon}(\mathbb{R}^n)$, $d, d^\perp \in \mathbb{S}^{n-1}$ *and* $\tau > 0$, *there is a CGO solution* w *as in* (4.1) *with*

$$\|r\|_{H^{s,p}} = O\left(\frac{1}{\tau^{n/p-s+\sigma}}\right).$$

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Remarks:

- Proven for $n = 2, 3$, $s \in \mathbb{R}$, and γ, ρ with regularity depending on n, s.
- Based on a result from [Päivärinta-Salo-Vesalainen '17].

Asymptotic analysis ($\tau \to \infty$)

$$\int_{\mathcal{C}_\varepsilon} (\gamma - 1)\nabla v \cdot \nabla w - k^2(\rho - 1) v w \, dx = \int_{\mathcal{K}_\varepsilon} \gamma \partial_\nu w (v - u) - w \left(\partial_\nu v - \gamma \partial_\nu u \right) \, ds$$

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- $$\left| \int_{\mathcal{C}_\varepsilon} \left[(\gamma(x) - 1) \nabla v(x) \cdot \nabla w(x) - \gamma_\beta(\hat{x}) V(\hat{x}) \cdot \eta |x|^{\alpha+\beta} e^{\eta \cdot x} \right] dx \right|$$
$$= \|\gamma_\beta V\|_{L^\infty(\mathcal{K})} O\big(\frac{1}{\tau^{n+\beta+\alpha}}\big) + O\big(\frac{1}{\tau^{n+\beta+\alpha-1+\sigma}}\big),$$

if $(\gamma(x) - 1)\gamma^{-1/2}(x) = \gamma_\beta(\hat{x})|x|^\beta + O(|x|^{\beta+\sigma})$ and
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- $$\left|\int_{\mathcal{C}_\varepsilon} \left[(\rho-1)vw - \rho_0(\hat{x})\tilde{v}(\hat{x})|x|^{\alpha_0+\beta_0}e^{\eta\cdot x}\right]dx\right|$$
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  if $(\rho(x)-1)\gamma^{-1/2}(x) = \rho_0(\hat{x})|x|^{\beta_0} + O(|x|^{\beta_0+\sigma})$ and $v(x) = \tilde{v}(\hat{x})|x|^{\alpha_0} + O(|x|^{\alpha_0+\sigma})$

A Contradiction

Lemma 2

Let $n = 2$ *(*$\beta_0 = 0$ *and* $\alpha_0 = N_0$*). There exist* $d \in \mathbb{S}^{n-1}$ *and* $C_{1,N_0} \neq 0$ *such that*

$$\int_{\mathcal{C}_\varepsilon} v_{N_0}(x) e^{\eta \cdot x}\, dx = C_{1,N_0} \tau^{-n-N_0} + o\left(\tau e^{-\varepsilon\tau/2}\right).$$

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$$\int_{\mathcal{C}_\epsilon} e^{\eta \cdot x} \tilde{V}_N \cdot \eta\, dx = C_0 \tau^{1-n-N} + o\left(\tau e^{-\epsilon\tau/2}\right),$$

where $C_0 \neq 0$ *unless* $\psi_0 = \frac{l\pi}{1+N} \in (0, \pi)$*, i.e.,* $N = \frac{\pi}{\psi_0} l - 1 \in \mathbb{N}$*, for some* $l \in \mathbb{N}$*.*

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Example: Corners of $c - 1$ with a jump while $a - 1$ vanish to the second order locally: $\beta = 2$, $\gamma_\beta \equiv 0$, $\alpha = N$, $\beta_0 = 0$, $\alpha_0 = N_0$,

$$\left|\int_{\mathcal{C}_\varepsilon} (\gamma - 1)\nabla v \cdot \nabla w - k^2(\rho - 1) v w\, dx\right| = O\Big(\frac{1}{\tau^{n+N+1+\sigma}}\Big) - k^2 \rho_0 \int_{\mathcal{C}_\varepsilon} v_{N_0}(x) e^{\eta\cdot x}\, dx$$

A local problem

Given a convex polyhedral cone $\mathcal{C}^{\varepsilon}_{x_0}$ with finitely many edges, consider

$$\begin{cases} \nabla \wedge \mathbf{E} - i\omega\mu_0\mathbf{H} = \mathbf{F}_1 & \text{in } \mathcal{C}^{\varepsilon}_{x_0}, \\ \nabla \wedge \mathbf{H} + i\omega\epsilon_0\mathbf{E} = \mathbf{F}_2 & \text{in } \mathcal{C}^{\varepsilon}_{x_0}, \\ \nu \wedge \mathbf{E} = \nu \wedge \mathbf{H} = 0 & \text{on } \partial\mathcal{C}^{\varepsilon}_{x_0} \setminus \partial B_{\varepsilon}(x_0), \end{cases} \tag{5.1}$$

A local problem

Given a convex polyhedral cone $\mathcal{C}_{x_0}^{\varepsilon}$ with finitely many edges, consider

$$\begin{cases} \nabla \wedge \mathbf{E} - i\omega\mu_0 \mathbf{H} = \mathbf{F}_1 & \text{in } \mathcal{C}_{x_0}^{\varepsilon}, \\ \nabla \wedge \mathbf{H} + i\omega\epsilon_0 \mathbf{E} = \mathbf{F}_2 & \text{in } \mathcal{C}_{x_0}^{\varepsilon}, \\ \nu \wedge \mathbf{E} = \nu \wedge \mathbf{H} = 0 & \text{on } \partial\mathcal{C}_{x_0}^{\varepsilon} \setminus \partial B_{\varepsilon}(x_0), \end{cases} \tag{5.1}$$

Theorem 3 (Blåsten-Liu-X. '19)

For any given $\mathbf{F}_j \in C^{\alpha}(B_{x_0}^{\varepsilon})^3$, $j = 1, 2$, *with* $\alpha \in (0, 1)$, *suppose there exists a solution* $(\mathbf{E}, \mathbf{H}) \in \left(H(\mathrm{curl}, \mathcal{C}_{x_0}^{\varepsilon})\right)^2$ *to* (5.1). *Then*

$$\mathbf{F}_1(x_0) = \mathbf{F}_2(x_0) = 0. \tag{5.2}$$

Sketch of the proof

To show $\mathbf{F}_0 = 0$, where $\mathbf{F}_2(x) = \mathbf{F}_0 + \tilde{\mathbf{F}}(x)$ with $|\tilde{\mathbf{F}}(x)| \leq C|x|^\alpha, \ x \in \mathcal{C}^\varepsilon$.

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An integral identity, $\nabla \wedge \mathbf{V} - i\omega\mu_0\mathbf{W} = 0$, $\nabla \wedge \mathbf{W} + i\omega\epsilon_0\mathbf{V} = 0$ in $\mathcal{C}^\varepsilon$,

$$\int_{\mathcal{C}^\varepsilon} (\mathbf{F}_1 \cdot \mathbf{W} + \mathbf{F}_2 \cdot \mathbf{V}) = \int_{\partial\mathcal{C}^\varepsilon} (\mathbf{W} \cdot (\nu \wedge \mathbf{E}) + \mathbf{V} \cdot (\nu \wedge \mathbf{H}))\,.$$

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To show $\mathbf{F}_0 = 0$, where $\mathbf{F}_2(x) = \mathbf{F}_0 + \tilde{\mathbf{F}}(x)$ with $|\tilde{\mathbf{F}}(x)| \leq C|x|^\alpha,\ x \in \mathcal{C}^\varepsilon$.

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Take $\mathbf{V}(x) = pe^{\rho\cdot x}$ and $\mathbf{W}(x) = \frac{1}{i\omega\mu_0}\rho \wedge pe^{\rho\cdot x}$ with

$$\rho/\tau = d + i\sqrt{1 + k^2/\tau^2}d^\perp \text{ and } p = d^\perp - i\sqrt{1 + k^2/\tau^2}d.$$

Sketch of the proof

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Asymptotics:

$$\int_{\mathcal{C}^\varepsilon} \left(\mathbf{F}_1 \cdot \mathbf{W} + \tilde{\mathbf{F}} \cdot \mathbf{V}\right) + \int_{\mathcal{C}\setminus\mathcal{C}^\varepsilon} \mathbf{F_0} \cdot \mathbf{V} = O\left(\tau^{-(3+\alpha)}\right).$$

Sketch of the proof

To show $\mathbf{F}_0 = 0$, where $\mathbf{F}_2(x) = \mathbf{F}_0 + \tilde{\mathbf{F}}(x)$ with $|\tilde{\mathbf{F}}(x)| \leq C|x|^\alpha,\ x \in \mathcal{C}^\varepsilon$.

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Asymptotics:

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Lemma 3

There exist $C > 0$, and $d, d^\perp \in \mathbb{S}^2$ such that $\lim_{\tau\to\infty} \mathbf{F}_0 \cdot p \neq 0$ and

$$\left|\int_{\mathcal{C}} e^{\rho\cdot x} dx\right| \geq C\tau^{-3}, \qquad \text{any } \tau \geq k.$$

Applications – sourse scattering

EM source scattering:

$$
\begin{aligned}
\nabla \wedge \mathbf{E}(x) - i\omega\mu_0 \mathbf{H}(x) &= \mathbf{J}_1(x), \quad x \in \mathbb{R}^3, \\
\nabla \wedge \mathbf{H}(x) + i\omega\epsilon_0 \mathbf{E}(x) &= \mathbf{J}_2(x), \quad x \in \mathbb{R}^3, \\
\lim_{|x|\to\infty} |x| \left(\sqrt{\mu_0}\mathbf{H} \times \frac{x}{|x|} - \sqrt{\varepsilon_0}\mathbf{E} \right) &= 0.
\end{aligned}
$$

Applications – sourse scattering

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$$\nabla \wedge \mathbf{H}(x) + i\omega\epsilon_0 \mathbf{E}(x) = \mathbf{J}_2(x), \quad x \in \mathbb{R}^3,$$
$$\lim_{|x|\to\infty} |x| \left(\sqrt{\mu_0}\mathbf{H} \times \frac{x}{|x|} - \sqrt{\varepsilon_0}\mathbf{E} \right) = 0.$$

Theorem 4 (Non-radiating sources with corners)

Let $\mathbf{J}_1$ *or/and* $\mathbf{J}_2$ *be a radiationless source with a corner* $(x_0, \mathcal{C}_{x_0}^{\varepsilon})$ *at its support. If* $\mathbf{J}_1|_{\mathcal{C}_{x_0}^{\varepsilon}} = \mathbf{F}|_{\mathcal{C}_{x_0}^{\varepsilon}}$ *with some* $\mathbf{F} \in C^{\alpha}(B_{x_0}^{\varepsilon})$ *and* $\alpha > 0$*, then*

$$\mathbf{J}_1(x_0) = 0.$$

Other applications

Shape determination of sources: Uniqueness in determining the polyhedral convex hull of the source by a single measurement.

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EM medium problem:

$$\begin{cases} \nabla \wedge \mathbf{E}^t - \mathrm{i}\omega\mu\mathbf{H}^t = 0, & \nabla \wedge \mathbf{H}^t + \mathrm{i}\omega\gamma\mathbf{E}^t = 0 & \text{in} \quad \Omega, \\ \nabla \wedge \mathbf{E}^0 - \mathrm{i}\omega\mu_0\mathbf{H}^0 = 0, & \nabla \wedge \mathbf{H}^0 + \mathrm{i}\omega\epsilon_0\mathbf{E}^0 = 0 & \text{in} \quad \Omega, \\ \nu \wedge \mathbf{E}^t = \nu \wedge \mathbf{E}^0, & \nu \wedge \mathbf{H}^t = \nu \wedge \mathbf{H}^0 & \text{on} \quad \partial\Omega. \end{cases}$$

Set $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}) := (\mathbf{E}^t, \mathbf{H}^t) - (\mathbf{E}^0, \mathbf{H}^0) \longrightarrow$ Source problem.

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Shape determination of sources: Uniqueness in determining the polyhedral convex hull of the source by a single measurement.

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Set $(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}) := (\mathbf{E}^t, \mathbf{H}^t) - (\mathbf{E}^0, \mathbf{H}^0) \longrightarrow$ Source problem.

- Corner scattering;
- Interior transmission eigenfunctions.

Further

- Concave corners;
- Anisotropic materials at the corner;
- Incident fields where no conclusion is made yet;
- Other dimensions;
- “Other shapes” scattering.

Thank you!