Linear stability of slowly rotating Kerr black holes

Peter Hintz (MIT)
with Dietrich Häfner (Grenoble) and András Vasy (Stanford)

Inverse Problems, Imaging and PDE
HKUST, May 22, 2019

Einstein vacuum equation

We are interested in the global behavior of solutions of

$$\mathrm{Ric}(g) = 0,$$

where g is a Lorentzian metric $(+---)$ on a 4-manifold M.

Here: study perturbations of special solutions.

Special solutions of $\mathrm{Ric}(g) = 0$

1. Minkowski space.

$$M = \mathbb{R}_t \times \mathbb{R}^3_x,$$
$$g_{(0,0)} = dt^2 - dx^2 = dt^2 - dr^2 - r^2 g_{\mathbb{S}^2}.$$

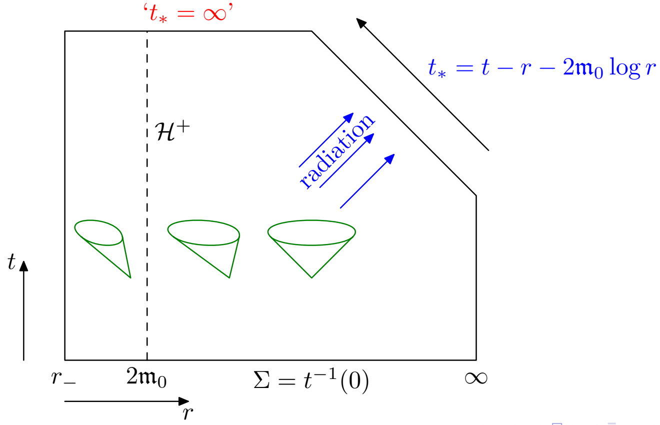
2. Schwarzschild black holes (mass $\mathfrak{m} > 0$).

$$M = \mathbb{R}_t \times (0,\infty)_r \times \mathbb{S}^2,$$
$$g_{(\mathfrak{m},0)} = \left(1 - \frac{2\mathfrak{m}}{r}\right) d\tilde{t}^2 - \left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1} dr^2 - r^2 g_{\mathbb{S}^2}$$
$$= g_{(0,0)} + \mathcal{O}(r^{-1}).$$

Illustration of the Schwarzschild metric

$$b_0 = (\mathfrak{m}_0, 0), \quad g_{b_0} = \left(1 - \frac{2\mathfrak{m}_0}{r}\right) d\tilde{t}^2 - \left(1 - \frac{2\mathfrak{m}_0}{r}\right)^{-1} dr^2 - r^2 g_{\mathbb{S}^2}.$$

Shown: $M = \mathbb{R}_t \times X, \quad X = [r_-, \infty) \times \mathbb{S}^2, \; r_- \in (0, 2\mathfrak{m}_0).$



Special solutions of $\mathrm{Ric}(g) = 0$, continued

3. Kerr black holes (mass $\mathfrak{m} > 0$, angular momentum $\mathfrak{a} \in \mathbb{R}^3$).

$$g_{(\mathfrak{m},\mathfrak{a})} = \cdots = g_{(\mathfrak{m},0)} + \mathcal{O}(r^{-2}).$$

   Consider slowly rotating Kerr black holes:

$$b := (\mathfrak{m}, \mathfrak{a}) \approx b_0 = (\mathfrak{m}_0, 0).$$

   on $M = \mathbb{R}_t \times X$.

g_b for such b is a smooth family of stationary metrics on M.

Initial value problem for $\mathrm{Ric}(g) = 0$

Given on $\Sigma = t^{-1}(0) \subset M$:

- γ: Riemannian metric on Σ,
- k: symmetric 2-tensor on Σ.

Find:

- Lorentzian metric g on M, $\mathrm{Ric}(g) = 0$,
- $\tau(g) := (-g|_{\Sigma}, \mathrm{II}^{\mathrm{g}}_{\Sigma}) = (\gamma, \mathrm{k})$.

Necessary and sufficient for local existence: constraint equations on (γ, k). (Choquet-Bruhat '52.)

Example

For $(\gamma, k) = (\gamma_b, k_b) := \tau(g_b)$, the solution of the initial value problem is g_b.

Kerr black hole stability conjecture

Conjecture

Given: $\alpha > 0$, *initial data* (γ, k) *on* Σ *such that*

$$\begin{aligned} |\gamma - \gamma_b| &\leq \epsilon r^{-1-\alpha}, \\ |k - k_b| &\leq \epsilon r^{-2-\alpha}, \end{aligned} \qquad \epsilon \ll 1.$$

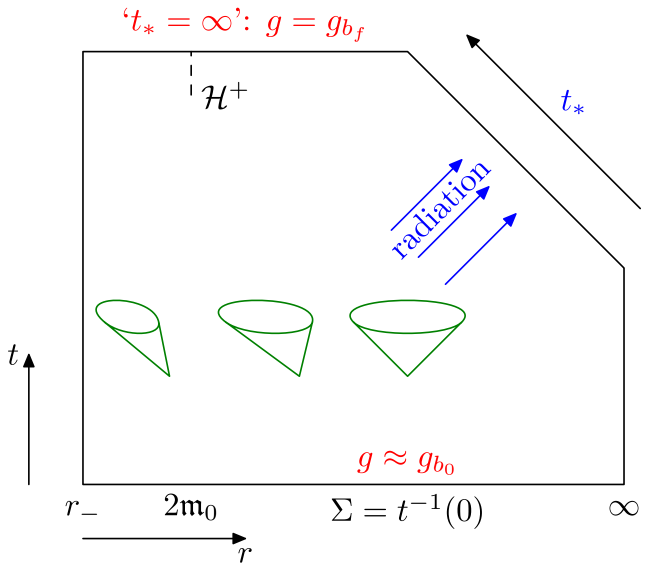
Then: *there exists a solution* g *of the initial value problem*

$$\mathrm{Ric}(g) = 0, \quad \tau(g) = (\gamma, k),$$

and parameters $b_f = (\mathfrak{m}_f, \mathfrak{a}_f)$ *such that*

$$g = g_{b_f} + \tilde{g}, \quad |\tilde{g}| \lesssim t_*^{-\beta}, \ \beta > 0.$$

H.–Vasy '18: Kerr–de Sitter black hole stability, $|\tilde{g}| \lesssim e^{-\beta t_*}$,
Klainerman–Szeftel '18: proof in special symmetry class.



'$t_* = \infty$': $g = g_{b_f}$
$\mathcal{H}^+$
t_*
radiation
$g \approx g_{b_0}$
t
r_-
$2\mathfrak{m}_0$
$\Sigma = t^{-1}(0)$
∞
r

$$g = g_{b_f} + \tilde{g}, \quad |\tilde{g}| \lesssim t_*^{-\beta}.$$

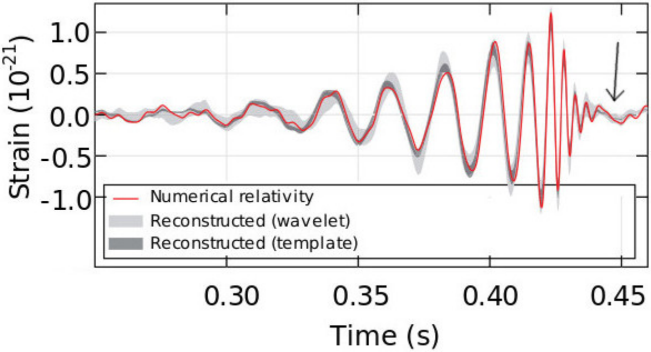


Figure: LIGO/Virgo Collaborations, PRL '16

Kerr stability: main issues

1. Find final black hole parameters $(\mathfrak{m}_f, \mathfrak{a}_f)$.
2. Diffeomorphism invariance: $\mathrm{Ric}(g) = 0 \Rightarrow \mathrm{Ric}(\Phi^* g) = 0$.
   - gauge fixing;
   - track location of black hole in chosen gauge.

In H.–Vasy: use Newton-type iteration scheme; naively $g_0 = g_{b_0}$,

$$\begin{cases} D_{g_0}\mathrm{Ric}(h_0) = -\mathrm{Ric}(g_0), \\ \text{initial data for } h_0 \text{ on } \Sigma, \end{cases} \implies \quad g_1 = g_0 + h_0, \quad \text{etc.}$$

Idea: read off improved guess of final black hole parameters and location/velocity from asymptotic behavior of h_0.

Linear stability (modulo gauge)

Consider black hole parameters $b \approx b_0 = (\mathfrak{m}_0, 0)$.

Theorem (Häfner–H.–Vasy '19)

Let γ', k' be symmetric 2-tensors on $\Sigma = t^{-1}(0)$ (satisfying the linearized constraint equations). Then there exists a symmetric 2-tensor h on M such that

$$D_{g_b}\mathrm{Ric}(h) = 0, \quad D_{g_b}\tau(h) = (\gamma', k'),$$

which decays to a linearized Kerr metric,

$$h = g'_b(b') + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-\beta}. \qquad \left(g'_b(b') := \frac{\mathrm{d}}{\mathrm{d}s} g_{b+sb'}\Big|_{s=0}. \right)$$

Gauge fixing

Eliminate diffeomorphism invariance: impose extra condition on g:

$$W(g) = \Box_{g,g_b}\mathbf{1} \; (= \text{1-form in } g,\, \partial g) = 0.$$

Then ('DeTurck trick'):

$$\begin{cases} \mathrm{Ric}(g) = 0, \\ W(g) = 0, \\ \text{initial data} \end{cases} \iff \text{IVP for } P(g) := \mathrm{Ric}(g) - \delta_g^* W(g) = 0.$$

Linearized version:

$$\begin{cases} D_{g_b}\mathrm{Ric}(h) = 0, \\ D_{g_b}W(h) = 0, \\ \text{initial data} \end{cases} \iff \text{IVP for } D_{g_b}P(h) \left(\approx \frac{1}{2}\Box_{g_b} h\right) = 0.$$

Main theorem

Let $L_b := D_{g_b} P$. Study $L_b h = 0$ with general initial data.

Theorem (Häfner–H.–Vasy '19)

Let $\alpha \in (0,1)$, and let $h_0, h_1 \in \mathcal{C}^\infty(\Sigma; S^2 T^*_\Sigma M)$,

$$|h_0| \lesssim r^{-1-\alpha}, \quad |h_1| \lesssim r^{-2-\alpha},$$

(and similar bounds for derivatives). Let

$$\begin{cases} L_b h = 0, \\ (h|_\Sigma,\ \mathcal{L}_{\partial_t} h|_\Sigma) = (h_0, h_1). \end{cases}$$

Then there exist $b' \in \mathbb{R} \times \mathbb{R}^3$ and $V \in$ fixed 6-dimensional space of vector fields on M such that

$$h = g'_b(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha}.$$

Main theorem, continued

$$h = g_b'(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha}.$$

Here,

$$V \in \operatorname{span}\{\text{asymptotic translations: } \partial_{x^i} + \mathcal{O}(r^{-1}), \\ \text{asymptotic boosts: } t\partial_{x^i} - x^i\partial_t + \text{l.o.t.}\}.$$

Can read off:

- change of black hole parameters,
- movement of black hole in chosen gauge.

Prior work

- Andersson–Bäckdahl–Blue–Ma '19: initial data with fast decay ($\alpha > 5/2$; then $b' = 0$)
- Dafermos–Holzegel–Rodnianski '16: $b = b_0$
- Johnson, Hung–Keller–Wang '16–'18: $b = b_0$
- Finster–Smoller '16

Strategy of proof

Recast as

$$L_b h = f.$$

Work on spectral side:

$$h(t_*, x) = \frac{1}{2\pi} \int_{\operatorname{Im}\sigma = C} e^{-i\sigma t_*} \widehat{L_b}(\sigma)^{-1} \hat{f}(\sigma, x)\, d\sigma.$$

Shift contour to $C = 0$. Precise description of resolvent:

$$\widehat{L_b}(\sigma)^{-1} = \sigma^{-2} R_2 + \sigma^{-1} R_1 + H^{1/2+\alpha},$$

where the range of R_1, R_2 is explicit (linearized Kerr, pure gauge).

Uses/is related to: Melrose, Vasy–Zworski, Vasy; Wunsch–Zworski, Dyatlov, H.; Vasy, Guillarmou–Hassell, Bony–Häfner.

Thank you!