Imaging Local Perturbations in Periodic Layered Media

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Joint work with

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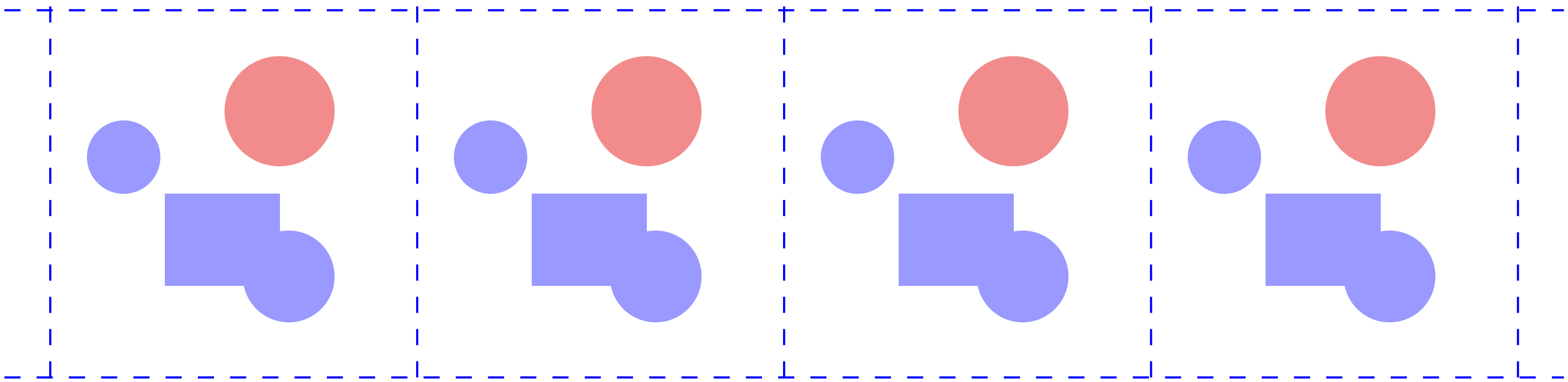
Scattering by a periodic layer



$$\nabla \cdot A \nabla u^s + k^2 n u^s = f \text{ for } (\tilde{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$$

u^s satisfies Rayleigh radiation condition

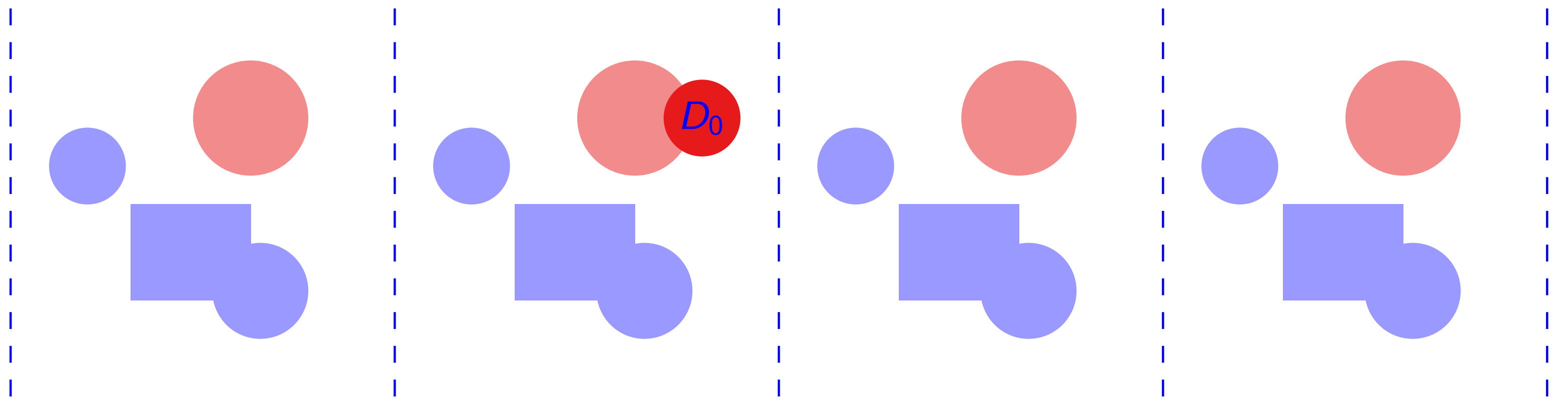
A, n are $\tilde{x}$-periodic L^∞-matrix valued function and scalar function, respectively.

$\mathrm{supp}(A - I)$ and $\mathrm{supp}(n - 1)$ is included in $\mathbb{R}^{d-1} \times (-h, h)$.

$f := \nabla \cdot (A - I)\nabla u^i + k^2(n - 1)u^i$ is quasi-periodic

T. Arens, G. Bao, L. Bourgeois, J. Elschner, S. Fliss, G. Hu, A. Kirsch, A. Lechleiter, J. Lin, P. Liu, T.P. Nguyen .....

The Problem

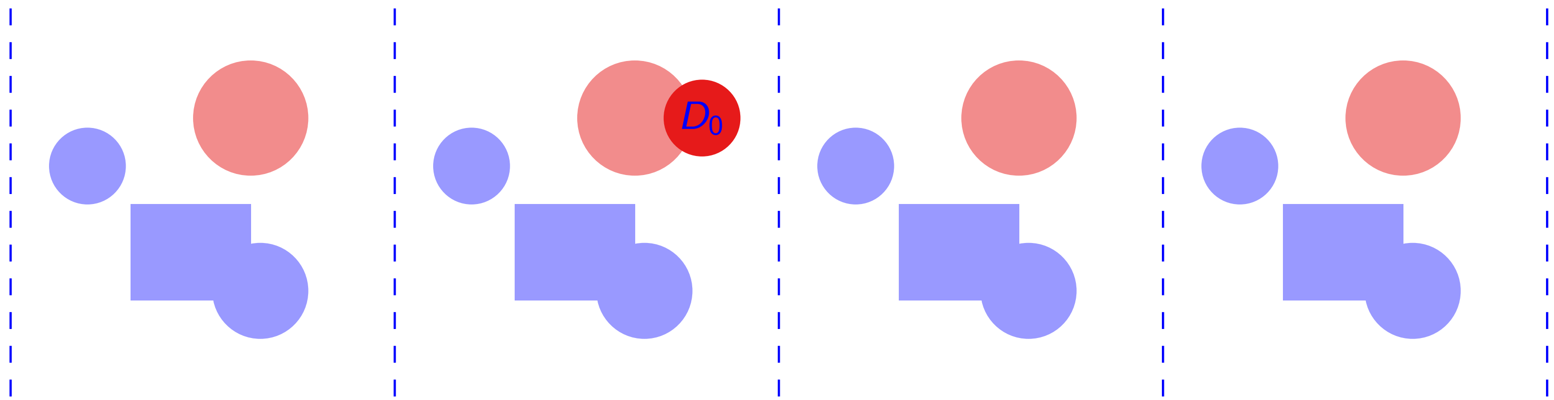


Reconstruct the support of a local perturbation D_0 in a periodic inhomogeneous layered medium from a knowledge of the scattered field due to several incident plane waves.

- Our main constraint is that a model for the periodic background is not available or otherwise it is hard to compute its Green's function.
- One cell consists of a collection of inhomogeneities situated in a known homogeneous medium and we do not want to reconstruct them

.

The Problem



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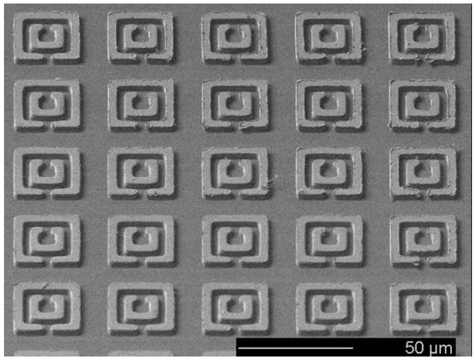
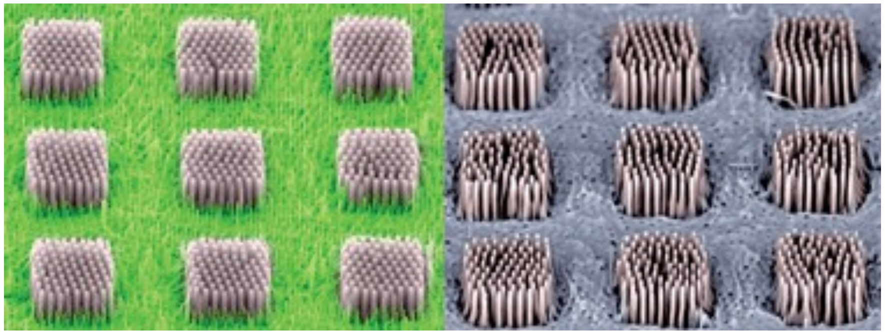
Some applications: Detection of local defects in periodic patterns!

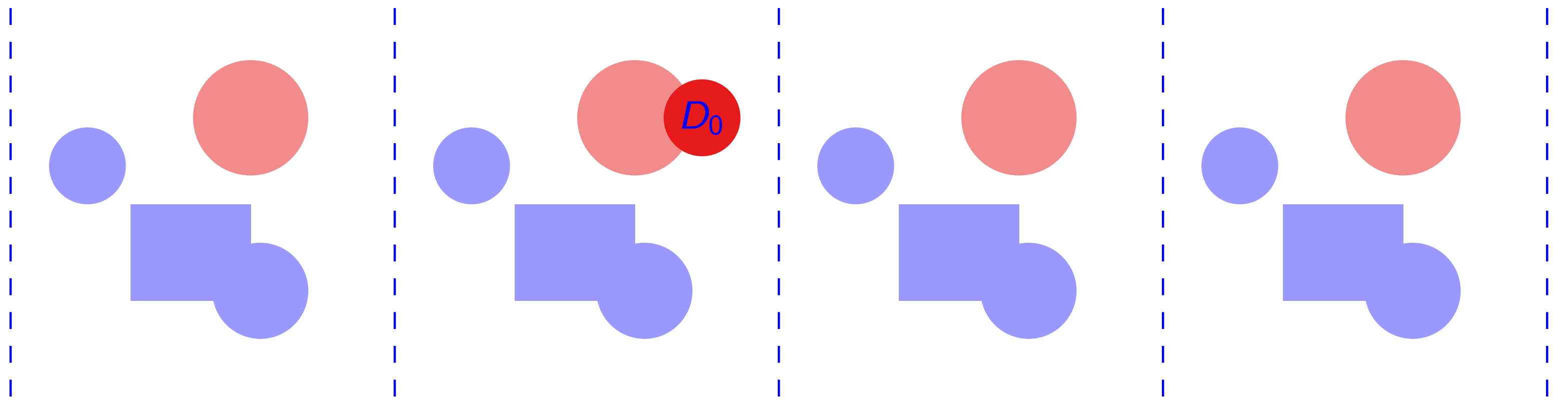


A meta-material



Nano-grass structures

Formulation of the problem



For sake of this presentation, we assume $A \equiv I$, $d = 2$.

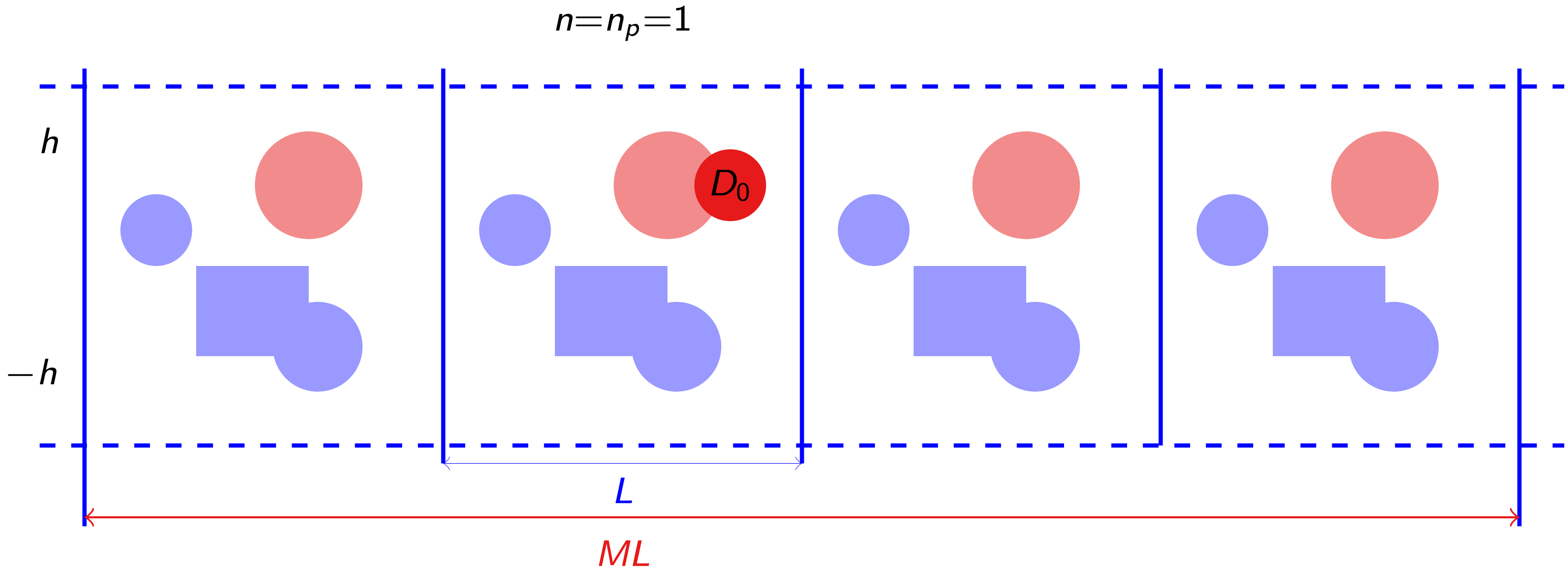
Denote by n_p the refractive index of periodic layer and by n the refractive index of the perturbed media, i.e. $\mathrm{supp}(n - n_p) = D_0$.

The unknown perturbation with support D_0 is located in one period which we don't know a priori. We assume to know only the period of the background.

Given measurements of the scattered wave at a fixed frequency

Recover the defect $D_0 := \mathrm{supp}(n - n_p)$ without knowing n and n_p

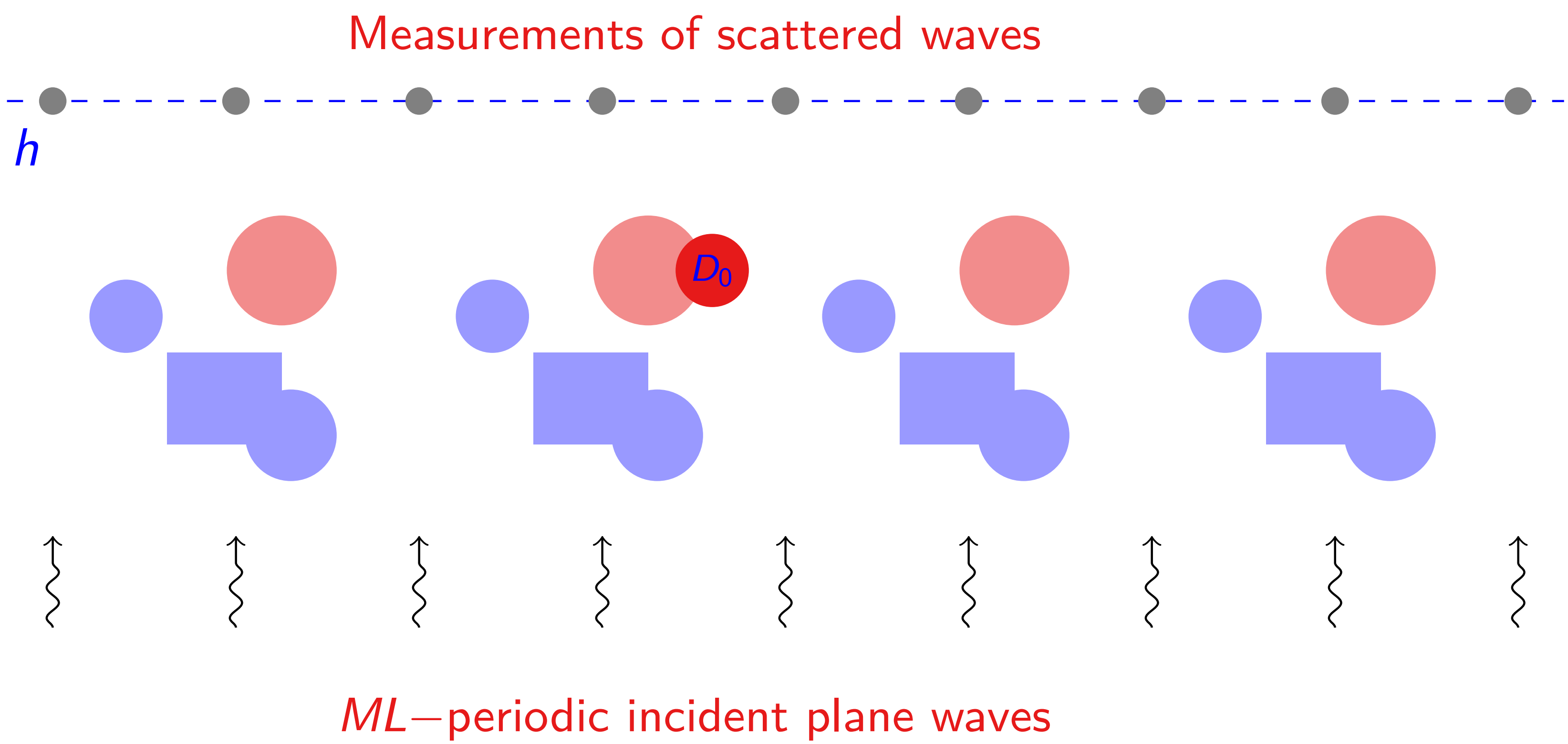
The approximate model problem



$n_p(\cdot, x_d)$ is $L-$periodic, $n = 1$ for $|x_d| > h > 0$ and n is $ML-$periodic.

- Consider a truncated domain of M periods, which contains the defect.
- Impose the ML-periodicity on index n.

The approximate model problem

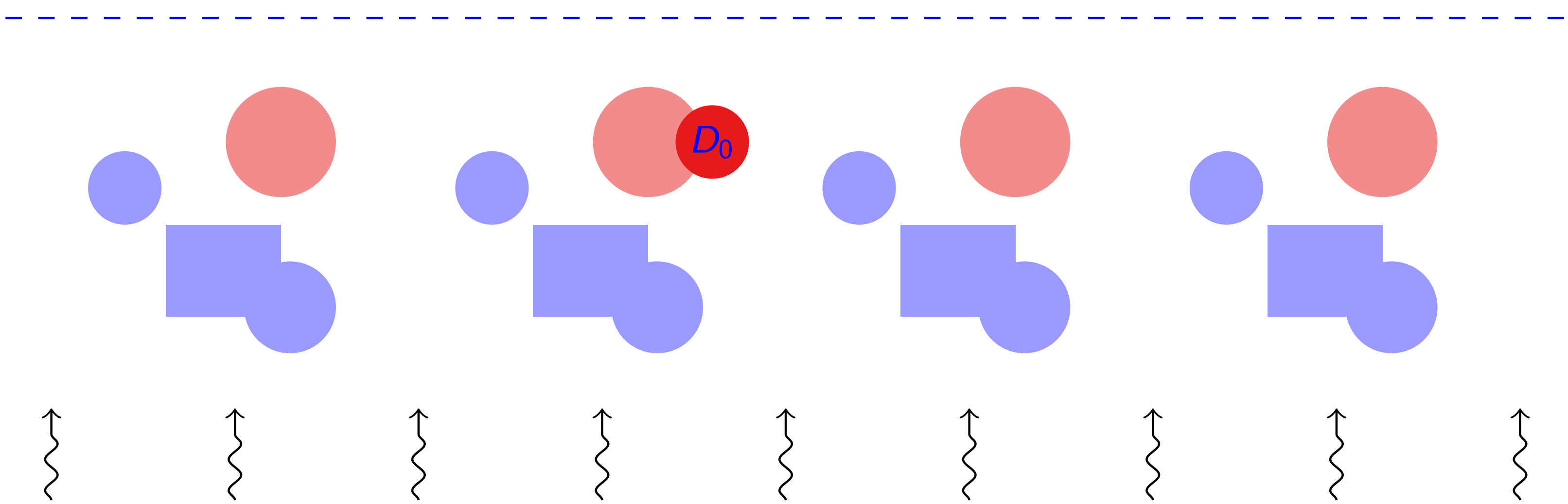


- Reconstruct $D_0 := \mathrm{supp}(n - n_p)$ from the given measurements of the scattered wave associated with $ML-$periodic incident waves

Description of the data

Total field ($ML-$periodic): $\Delta u + k^2 n u = 0$ in $\mathbb{R}^2$, $\quad k \in \mathbb{R}$

Scattered wave: $u_j^s(x) = \sum_\ell \widehat{u}_j^s(\ell) e^{\mathrm{i}(\alpha_\ell x_1 + \beta_\ell (x_2 - h))}$, $x_2 > h$



$ML-$periodic incident plane waves: $u_j^i = \frac{1}{\overline{\beta_j}} e^{\mathrm{i}(\alpha_j x_1 + \overline{\beta_j}(x_2 - h))} \quad j \in \mathbb{Z}$

$$\alpha_j := 2\pi j/(ML), \quad \beta_j := \sqrt{k^2 - \alpha_j^2}, \quad \mathrm{Im}\,\beta_j \geq 0$$

Scattered field: $\quad \Delta u^s + k^2 n u^s = k^2(1 - n)u^i$

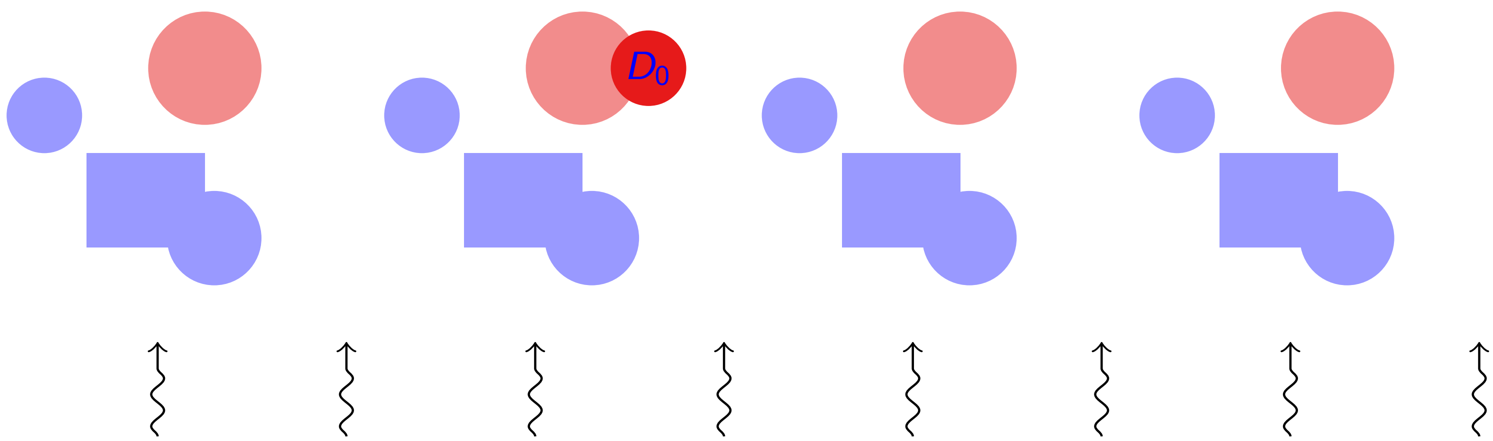
+ Radiation condition

Description of the data

Data for the inverse problem:

The measurements are given by the Rayleigh sequences $\{\widehat{u}_j^s(\ell)_{\ell\in\mathbb{Z}}\}$

$$[\cdots,\widehat{u}_j^s(\ell-M),\cdots,\widehat{u}_j^s(\ell-1),\widehat{u}_j^s(\ell),\widehat{u}_j^s(\ell+1),\widehat{u}_j^s(\ell+2)\cdots,\widehat{u}_j^s(\ell+M),\cdots,]$$



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$$\alpha_j:=2\pi j/(ML),\quad \beta_j:=\sqrt{k^2-\alpha_j^2},\quad \mathrm{Im}\,\beta_j\geq 0$$

Scattered field: $\quad \Delta u^s+k^2 n u^s=k^2(1-n)u^i$

$$+\text{ Radiation condition}$$

The near field operator N

From the measurements given by the Rayleigh sequence:

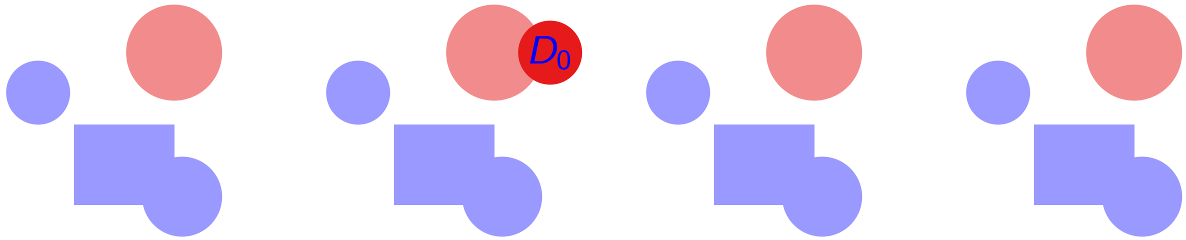
$$[\cdots, \widehat{u}_j^s(\ell - M), \cdots, \widehat{u}_j^s(\ell - 1), \widehat{u}_j^s(\ell), \widehat{u}_j^s(\ell + 1), \widehat{u}_j^s(\ell + 2) \cdots, \widehat{u}_j^s(\ell + M), \cdots,]$$

we can define the **near field operator:** $N : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$N(a) := \Big[\sum_{j\in\mathbb{Z}} \widehat{u}_j^s(\ell) a(j)\Big]_{\ell\in\mathbb{Z}}$$

Standard ITP

Linked to the analysis of the operator N



Domain $D := \text{supp}(1 - n)$

Factorization of the operator N: $N = GH$, $N = H^* TH$

- $H : \ell^2(\mathbb{Z}) \to L^2(D)$ is the Herglotz operator defined by

$$H(a) := \sum_{j \in \mathbb{Z}} a(j) u_j^i$$

$$\overline{\mathcal{R}(H)} := \{v \in L^2(D) : \Delta v + k^2 v = 0, \text{ in } D$$

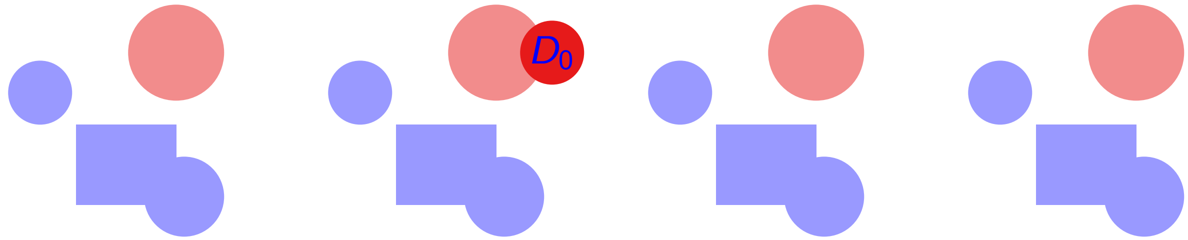
- $G : \overline{\mathcal{R}(H)} \to \ell^2(\mathbb{Z})$ is the solution operator defined by

$$Gv := \{\widehat{w}(\ell)\}_\ell \text{ the Rayleigh coefficients of } w$$

with w: $\Delta w + nk^2 w = k^2(1 - n)v$ in $\mathbb{R}^2$ + Radiation condition.

Standard ITP

Linked to the analysis of the operator N



Domain $D := \text{supp}(1 - n)$

Injectivity of the operator G is linked with the TEP

$(u, v) \in L^2(D) \times L^2(D)$ such that $w := u - v \in H^2(D)$ and

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ (u - v) = 0 & \text{on } \partial D, \\ \frac{\partial}{\partial \nu}(u - v) = 0 & \text{on } \partial D. \end{cases}$$

k is a transmission eigenvalue if this problem has non-trivial solution.

F. Cakoni, D. Colton and H. Haddar (2016), CBMS-NSF

Standard ITP

Denote by $\phi^z \in \ell^2(\mathbb{Z})$ the Rayleigh sequence of $\Phi(\cdot, z)$ - *ML* periodic

$$\Delta\Phi(\cdot, z) + k^2\Phi(\cdot, z) = -\delta_z.$$

- If k is not a transmission eigenvalue then

  G is injective and $N = GH$ has dense range.

- Moreover $\phi^z \in \mathrm{Range}(G)$ if and only if $z \in D$.

If $z \in D$ then $G(v_z) = \phi^z$ if and only if $(u_z, v_z) \in L^2(D) \times L^2(D)$ such that $u_z - v_z \in H^2(D)$ and satisfy the

Interior Transmission Problem (ITP):

$$\begin{cases} \Delta u_z + k^2 n u_z = 0 \quad \text{in } D, \\ \Delta v_z + k^2 v_z = 0 \quad \text{in } D, \\ (u_z - v_z) = \Phi(\cdot, z) \quad \text{on } \partial D, \\ \dfrac{\partial}{\partial \nu}(u_z - v_z) = \dfrac{\partial}{\partial \nu}\Phi(\cdot, z) \quad \text{on } \partial D. \end{cases} \tag{1}$$

Factorization

- **Define**

$$N_\sharp := |\mathrm{Re}\, N| + |\mathrm{Im}\, N|$$

- **The factorization of the near field operator $N_\sharp$**

  Assume that $(n-1)^{-1} \in L^\infty(D)$ and $\mathrm{Re}\,(n) - 1$ or $1 - \mathrm{Re}\,(n)$ is positive definite in a neighborhood of ∂D. Then the following factorization holds:

$$N_\sharp = H^*\, T_\sharp\, H,$$

  where $T_\sharp : L^2(D) \to L^2(D)$ is self-adjoint and coercive on $\overline{\mathcal{R}(H)}$.

- $\mathrm{Range}(N_\sharp^{1/2}) = \mathrm{Range}(G)$

$$\phi^z \in \mathrm{Range}(N_\sharp^{1/2}) \iff z \in D$$

Generalized Linear Sampling Method

Consider the functional:

$$J_\alpha(\phi^z; a) := \alpha(N_\sharp a, a) + \|Na - \phi^z\|^2 \quad \forall a \in \ell^2(\mathbb{Z}).$$

Let $a^{z,\alpha} \in \ell^2(\mathbb{Z})$, $\alpha > 0$, be a minimizing sequence such that

$$J_\alpha(\phi^z; a^{z,\alpha}) \leq \inf_{a \in \ell^2(\mathbb{Z})} J_\alpha(\phi^z; a) + o(\alpha) \text{ as } \alpha \to 0.$$

Then,

$$\phi^z \in \text{Range}(G) \text{ or } z \in D \iff \lim_{\alpha \to 0} (N_\sharp a^{z,\alpha}, a^{z,\alpha}) < \infty.$$

Moreover, if $\phi^z \in \text{Range}(G)$ then $Ha_{z,\alpha} \to v_z$ as $\alpha \to 0$ such that $G(z) = \phi^z$.

The single Floquet-Bloch near field operator

Fix $q \in \{0, \dots, M-1\}$. The measurements are formed by only the $q + \ell M$ Rayleigh coefficients

$$[\cdots, \widehat{u}_j^s(q-M), \cdots, \widehat{u}_j^s(q-1), \widehat{u}_j^s(q), \widehat{u}_j^s(q+1), \widehat{u}_j^s(q+2) \cdots, \widehat{u}_j^s(q+M), \cdots,$$

Single Floquet-Bloch mode near field operator: $N_q : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$:

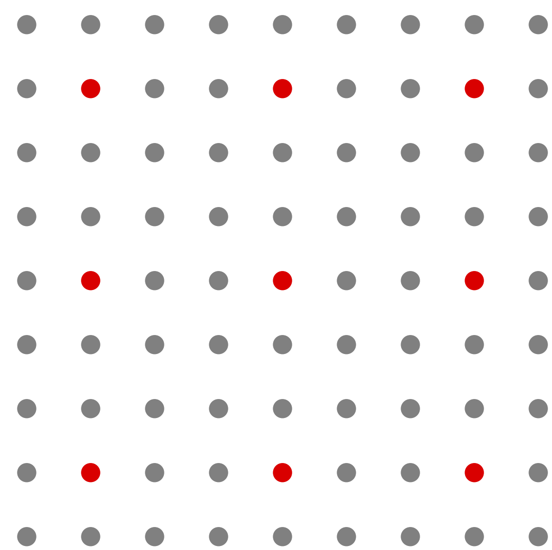
$$N_q(a) := \Big[\sum_{j \in \mathbb{Z}} \widehat{u}^s_{q+jM}(q + \ell M) a(j) \Big]_{\ell \in \mathbb{Z}}$$

N_q is associated with α_q-quasi periodic fields where $\alpha_q = \frac{2\pi}{ML} q$.

Relationship between N and N_q

$$N_q = I_q^* \, N \, I_q$$

where $(I_q a)(\ell) = \begin{cases} a(j) & \text{if } \ell = q + jM \\ 0 & \text{otherwise} \end{cases}$



"sub-matrix" N_q in red

Properties of N_q

Thus $N_q = (HI_q)^* T(HI_q) = H_q^* T H_q = \left(H_q^* T\right) H_q = G_q H_q$

- $H_q : \ell^2(\mathbb{Z}) \to L^2(D)$,

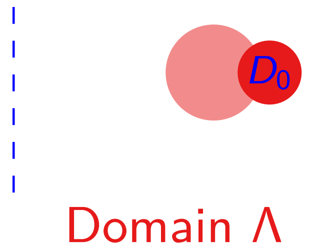
$$\overline{\mathcal{R}(H_q)} := \{v \in L^2(D) :\ \Delta v + k^2 v = 0,\ v|_{D_p} \text{is } \alpha_q\text{-quasi-periodic}\}.$$

- $G_q := I_q^* G : \overline{\mathcal{R}(H_q)} \to \ell^2(\mathbb{Z})$ and $G_q : v \mapsto I_q^* \{\widehat{w}_q(\ell)\}_{\ell \in \mathbb{Z}}$

$$\Delta w_q + k^2 n w_q = k^2(n_p - n)(w - w_q) + k^2(1 - n)v \text{ in } \Omega_0,$$

where Ω_0 defective period w_q is radiating α_q-quasi-periodic, with w

$$\Delta w + n k^2 w = k^2(1 - n)v \text{ in } \mathbb{R}^2 + \text{Radiation condition}$$



G_q is injective if and only the problem: Find $w_q \in H_0^2(\Lambda)$ and $v \in L^2(\Lambda)$ such that

$$\Delta w_q + k^2 n w_q = k^2(n_p - n)(w - w_q) + k^2(1 - n)v \text{ in } \Lambda$$

$$\Delta v + k^2 v = 0 \text{ in } \Lambda$$

and

$$\Delta w + n k^2 w = k^2(1 - n)v \text{ in } \mathbb{R}^2 + \text{Radiation condition}$$

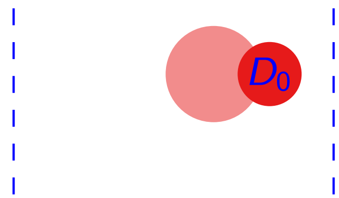
has nontrivial solution.

This is a new transmission eigenvalue problem!

The goal is to express $w - w_q$ in terms of v.

The new ITP

Linked with the analysis of the operator N_q



$(u, v) \in L^2(\Lambda) \times L^2(\Lambda)$ such that $w_q := u - v \in H^2(\Lambda)$ and

$$\begin{cases} \Delta u + k^2 n u = k^2(n_p - n) S_k(v) & \text{in } \Lambda, \\ \Delta v + k^2 v = 0 & \text{in } \Lambda, \\ (u - v) = \varphi & \text{on } \partial\Lambda, \\ \frac{\partial}{\partial \nu}(u - v) = \psi & \text{on } \partial\Lambda. \end{cases}$$

$$S_k : v \mapsto \int_\Lambda k^2(1 - n_p) v(y) \Big(\sum_{0 \neq m \in \mathbb{Z}_M} e^{\mathrm{i}\alpha_q m L} \Phi(n_p; x - mL - y) \Big) \, \mathrm{d}y \,,$$

$\Phi(n_p; \cdot)$ is the ML-periodic fundamental solution given by

$$\Delta \Phi(n_p; \cdot) + k^2 n_p \Phi(n_p; \cdot) = -\delta_0$$

Analysis of the new ITP

Property of the operator S_k

$$u \in H_0^2(\Lambda) \text{ and } f \in L^2(\Lambda),$$

$$\begin{cases} \Delta u + k^2 n u = (1-n)v + (n_p - n)S_k(v) + F & \text{in} \quad \Lambda, \\ \Delta v + k^2 v = 0 & \text{in} \quad \Lambda. \end{cases}$$

Property of the operator S_k: We can prove that there exists $\theta > 0$ and $C > 0$ such that

$$\|S_{i\kappa}(v)\|_{L^2(\Lambda)} \leq Ce^{-\theta\kappa}\|v\|_{L^2(\Lambda)}, \quad \kappa > 0, \ \forall v \in L^2(\Lambda)$$

and $S_k - S_{i\kappa}$ is compact.

Theorem

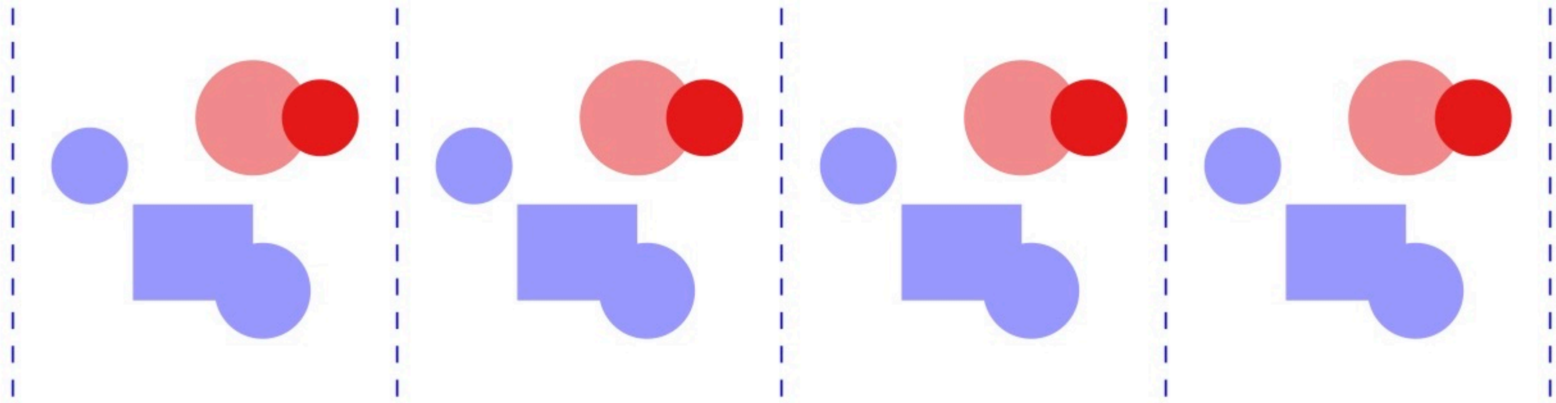
If $\mathrm{Re}\,(n-1) > 0$ or $\mathrm{Re}\,(1-n) > 0$ uniformly in a neighborhood of $\partial\Lambda$ from inside, the new interior transmission problem is Fredholm of index 0. The new transmission eigenvalues form at most a discrete set with $+\infty$ as the only possible accumulation point.

Factorization of $N_{q,\sharp}$

Let $N_{q,\sharp} = |\mathrm{Re}\, N_q| + |\mathrm{Im}\, N_q|$.

- Then $N_{q,\sharp} = H_q^* \, T_\sharp \, H_q$ where $T_\sharp : L^2(D) \to L^2(D)$ is self-adjoint and coercive on $\overline{\mathrm{Range}(H_q)}$.
- $\mathrm{Range}(N_{q,\sharp}^{1/2}) = \mathrm{Range}(G_q)$.
- Let $\phi_q^z \in \ell^2(\mathbb{Z})$ be the Rayleigh sequence associated with $\Phi_q(\cdot, z)$ α_q−quasi-periodic solution of $\Delta\Phi(\cdot, z) + k^2\Phi(\cdot, z) = -\delta_z$. Then

$$I_q^* \phi_q^z \in \mathrm{Range}(G_q) \iff z \in \widehat{D}_p$$



Description of the Algorithm

The algorithm uses:

- $N_\sharp := |\mathrm{Re}\,(N)| + |\mathrm{Im}\,(N)|, \quad N_{q,\sharp} := |\mathrm{Re}\,(N_q)| + |\mathrm{Im}\,(N_q)|$
- $\phi^z \in \ell^2(\mathbb{Z})$ the Rayleigh sequence associated with the $\Phi(\cdot, z)$ - *ML* periodic solution of

$$\Delta\Phi(\cdot, z) + k^2\Phi(\cdot, z) = -\delta_z \qquad (*)$$

- $\phi_q^z \in \ell^2(\mathbb{Z})$ the Rayleigh sequence associated with $\Phi_q(\cdot, z)$- α_q−quasi-periodic solution of $(*)$

Define

$$J_\alpha(\phi; a) := \alpha(N_\sharp a, a) + \|Na - \phi\|^2$$

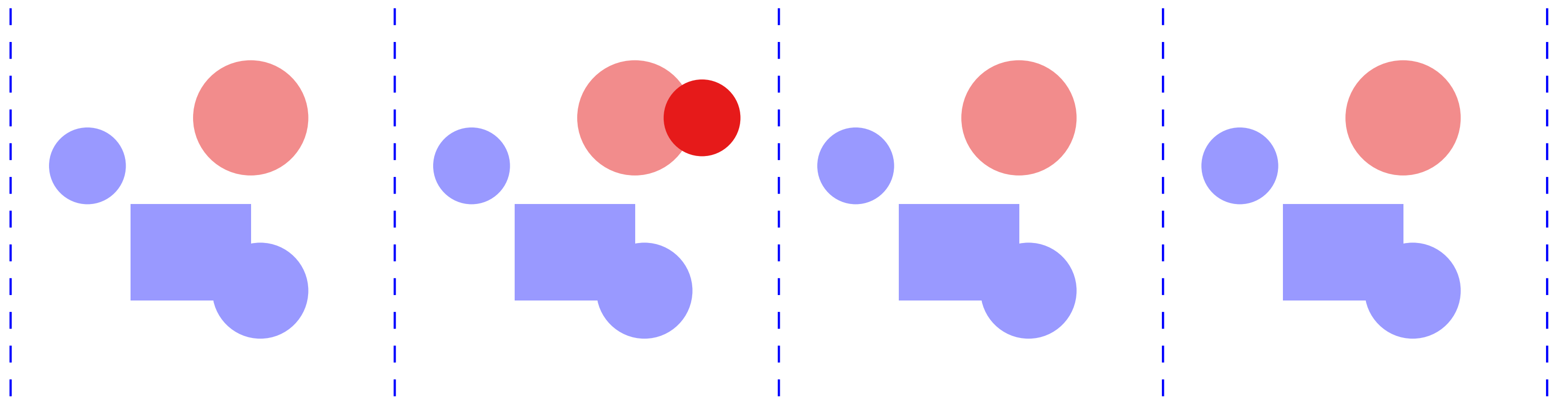
$$J_{q\alpha}(\phi; a) := \alpha(N_{q,\sharp} a, a) + \|N_q a - \phi\|^2$$

$a^{z,\alpha} \in \ell^2(\mathbb{Z})$ minimizing sequence for $J_\alpha(\phi^z; a)$

$a_q^{z,\alpha} \in \ell^2(\mathbb{Z})$ minimizing sequence for $J_\alpha(\phi_q^z; a)$

$\tilde{a}^{z,\alpha} \in \ell^2(\mathbb{Z})$ minimizing sequence for $J_{\alpha,q}(I_q^* \phi_q^z; a)$

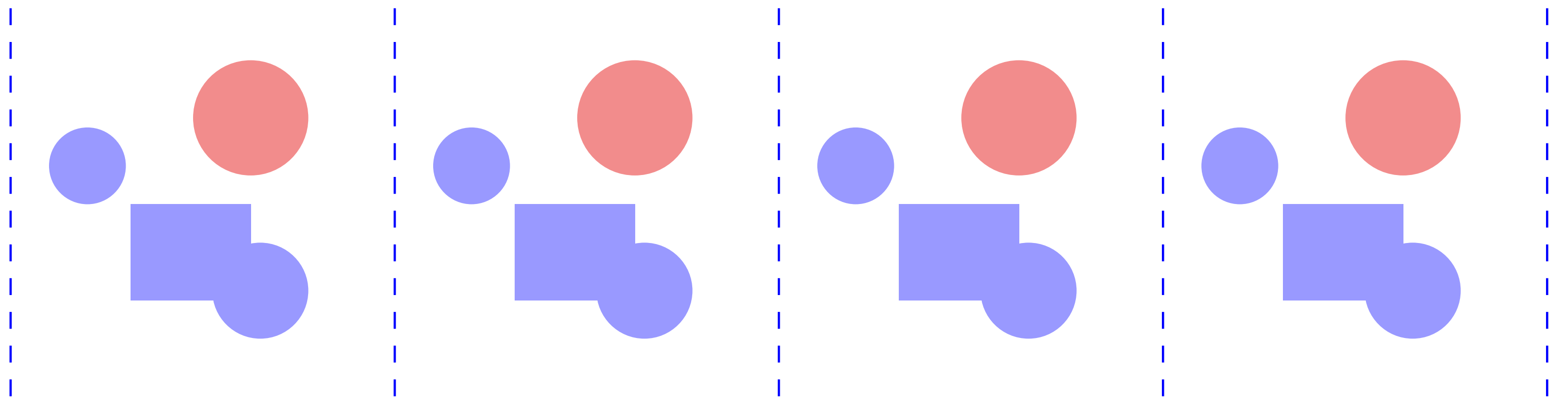
Indicator function



If the standard ITP is well-posed then:

- $z \in D$ iff $\lim_{\alpha \to 0}(N_\sharp a^{\alpha,z}, a^{\alpha,z}) < \infty, \qquad D = \mathrm{supp}(n-1)$

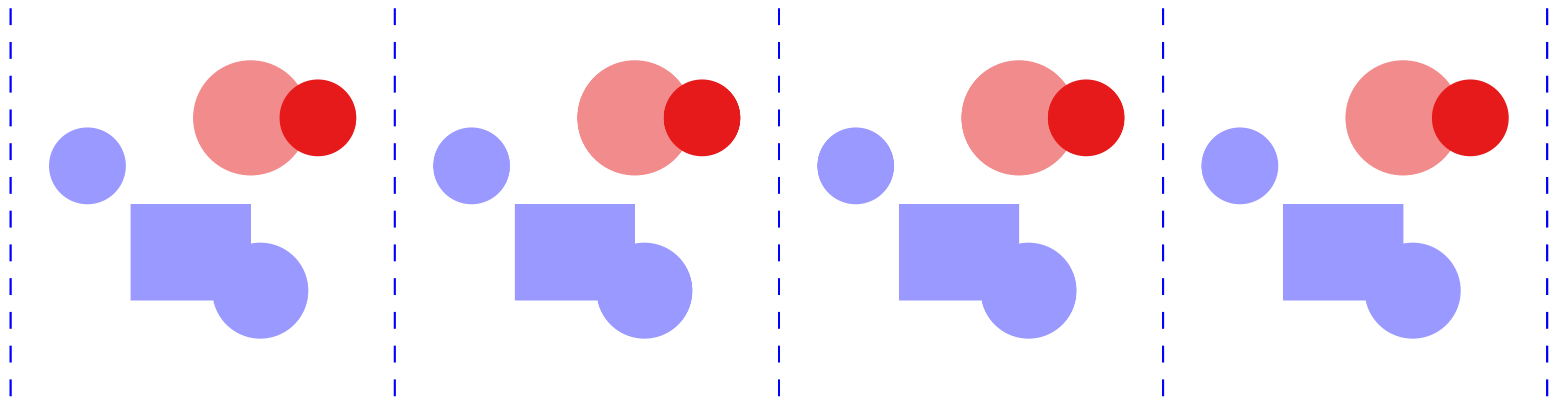
Indicator function



If the standard ITP is well-posed then:

- $z \in D$ iff $\lim_{\alpha\to 0}(N_\sharp a^{\alpha,z}, a^{\alpha,z}) < \infty$, $D = \mathrm{supp}(n-1)$
- $z \in D_p$ iff $\lim_{\alpha\to 0}(N_\sharp a_q^{\alpha,z}, a_q^{\alpha,z}) < \infty$, $D_p = \mathrm{supp}(n_p-1)$

Indicator function



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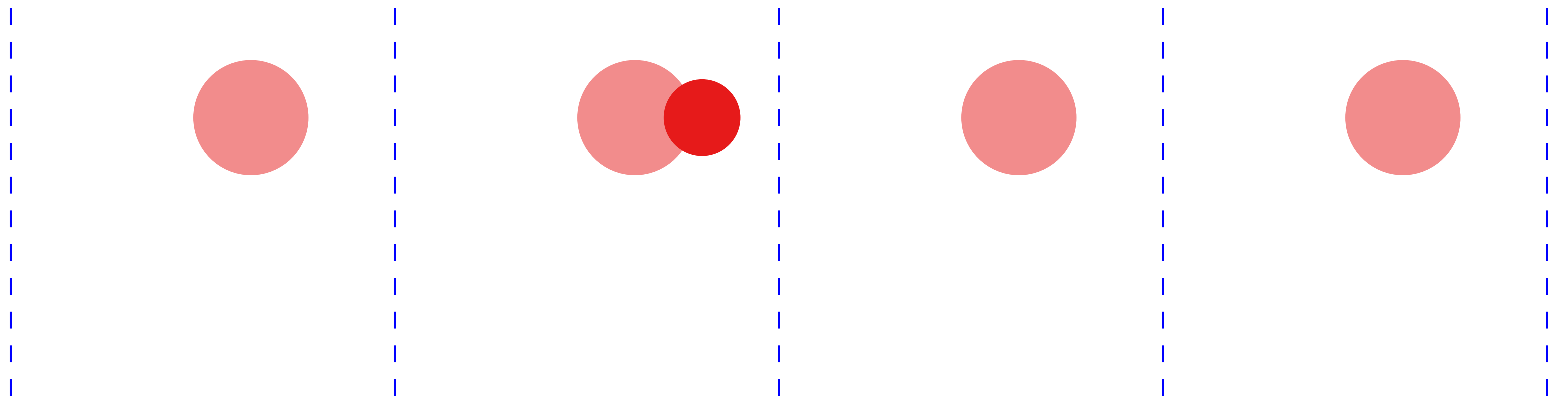
- $z \in D$ iff $\lim_{\alpha\to 0}(N_\sharp a^{\alpha,z}, a^{\alpha,z}) < \infty$, $\quad D = \mathrm{supp}(n-1)$
- $z \in D_p$ iff $\lim_{\alpha\to 0}(N_\sharp a_q^{\alpha,z}, a_q^{\alpha,z}) < \infty$, $\quad D_p = \mathrm{supp}(n_p-1)$

If the new ITP is well-posed then:

- $z \in \widehat{D}_p$ iff $\lim_{\alpha\to 0}(N_{q,\sharp} I_q \tilde{a}_q^{\alpha,z}, I_q \tilde{a}_q^{\alpha,z}) < \infty$

  where $z \in \widehat{D}_p$ is the support of L-periodic extension of defective cell.

Indicator function



Theorem: (Reconstruction of defect): Define

$$\mathcal{I}_\alpha(z) = (N_\sharp a^{\alpha,z}, a^{\alpha,z}) \left(1 + \frac{(N_\sharp a^{\alpha,z}, a^{\alpha,z})}{D_\alpha(z)}\right).$$

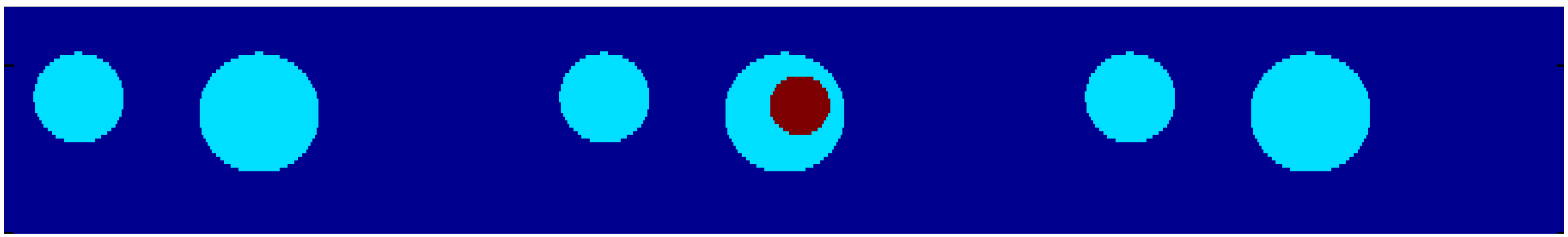
$$D_\alpha(z) = \left(N_\sharp(a_q^{\alpha,z} - I_q \tilde{a}_q^{\alpha,z}), (a_q^{\alpha,z} - I_q \tilde{a}_q^{\alpha,z})\right).$$

Then $z \in D_0 \cup \mathcal{O}_p$ iff $\lim_{\alpha \to 0} \mathcal{I}_\alpha(z) < \infty.$

provided the forward problem, the standard and the new ITP are well-posed. (Note that $D_\alpha(z) \le \| T_\sharp \| \| H a_q^{\alpha,z} - H_q a_q^{\alpha,z} \|^2_{L^2(D)}$)

Some numerical results

- The parameters of periodic background:
  - $k = \pi/3.14$, $n_p = 2$ inside the discs of radii r_1 and r_2, $n_p = 1$ otherwise
  - $L = 2\pi$, $h = 1.5L/k$, $r_1 = 0.3L/k$, $r_2 = 0.4L/k$



Exact geometry

- $n = 4$ inside $r_w := 0.25L/k$.
- Index of incident plane waves:

$$j = q + \ell M, \ -\frac{M}{2} + 1 \leq q \leq \frac{M}{2}, \ N_{min} \leq \ell \leq N_{max}$$

- $M = 3$, $N_{min} = -5$, $N_{max} = 5$
- 1% added multiplicative random noise

Some numerical results

- The parameters of periodic background:
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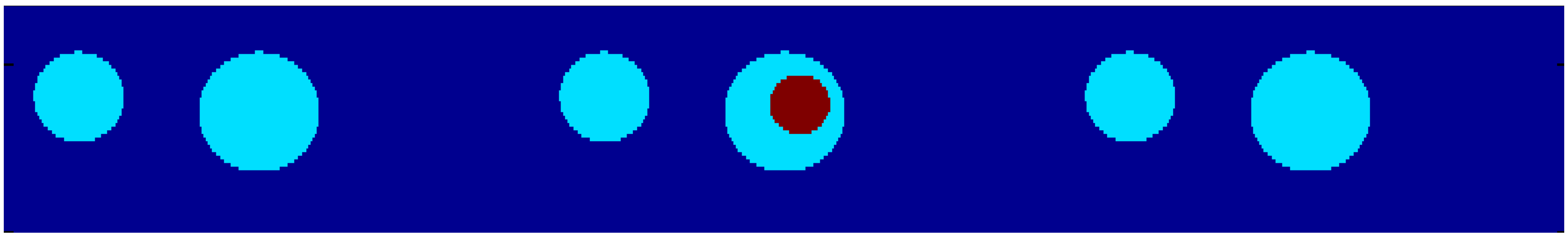
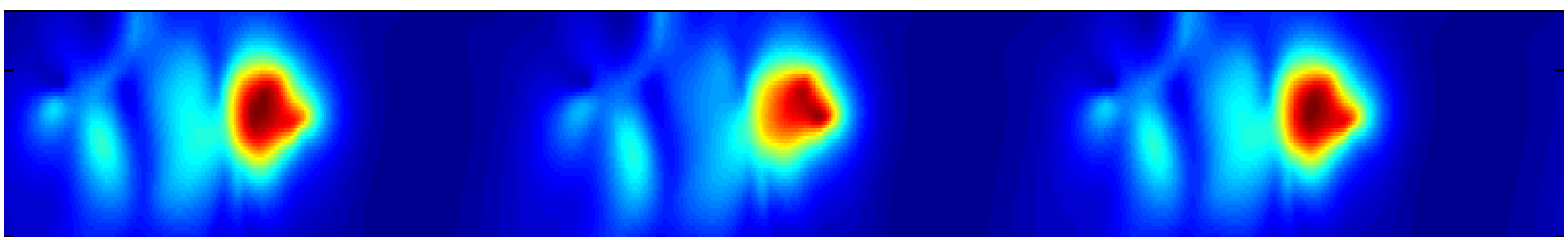
  - $L = 2\pi$, $h = 1.5L/k$, $r_1 = 0.3L/k$, $r_2 = 0.4L/k$



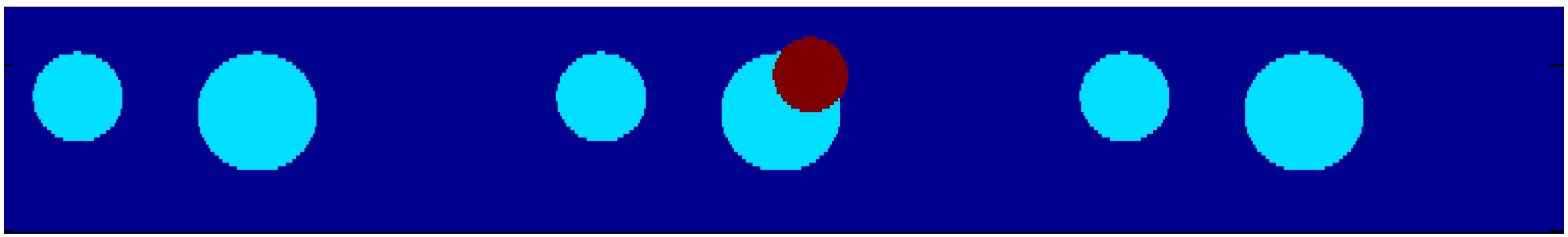
Exact geometry



Indicator function $z \mapsto \mathcal{I}_\alpha(z)$

Some numerical results

- The parameters of periodic background:
  - $k = \pi/3.14$, $n_p = 2$ inside the discs of radii r_1 and r_2, $n_p = 1$ otherwise
  - $L = 2\pi$, $h = 1.5L/k$, $r_1 = 0.3L/k$, $r_2 = 0.4L/k$



Exact geometry

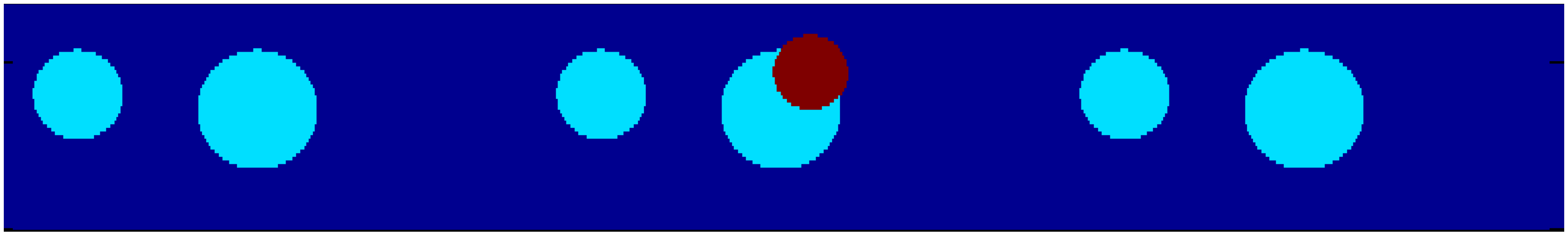
- $n = 4$ inside $r_w := 0.25L/k$.
- Index of incident plane waves:

$$j = q + \ell M,\ -\frac{M}{2} + 1 \leq q \leq \frac{M}{2},\ N_{min} \leq \ell \leq N_{max}$$

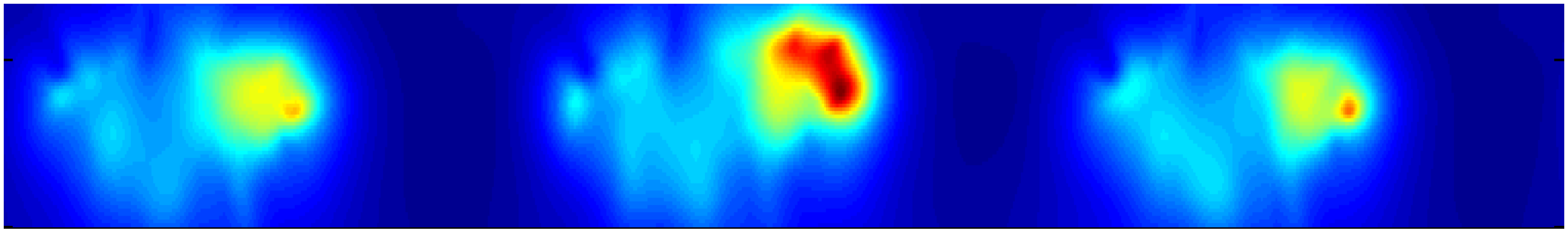
- $M = 3$, $N_{min} = -5$, $N_{max} = 5$
- 1% added multiplicative random noise

Some numerical results

- The parameters of periodic background:
  - $k = \pi/3.14$, $n_p = 2$ inside the discs of radii r_1 and r_2, $n_p = 1$ otherwise
  - $L = 2\pi$, $h = 1.5L/k$, $r_1 = 0.3L/k$, $r_2 = 0.4L/k$



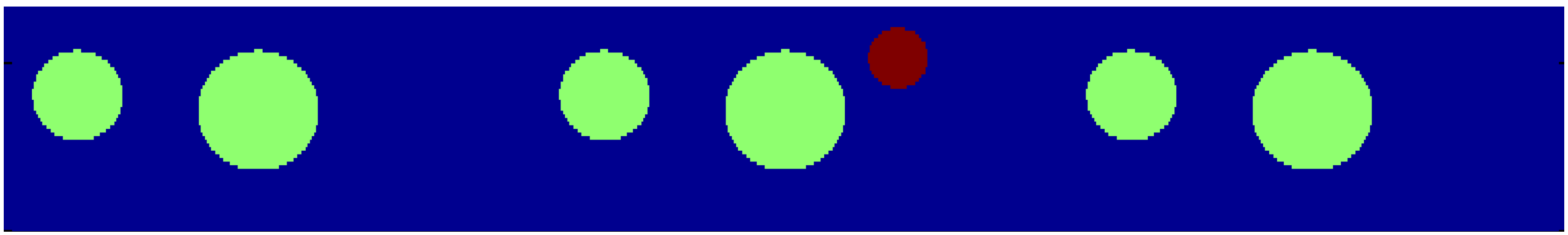
Exact geometry



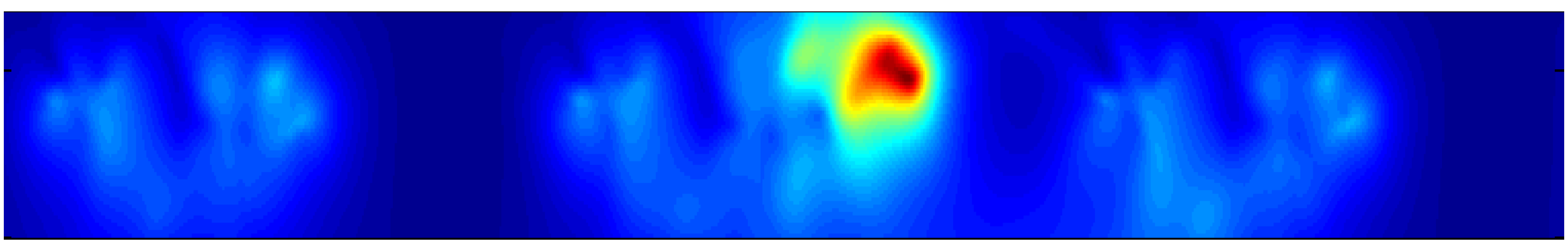
Indicator function $z \mapsto \mathcal{I}_\alpha(z)$

Some numerical results

- The parameters of periodic background:
  - $k = \pi/3.14$, $n_p = 2$ inside the discs of radii r_1 and r_2, $n_p = 1$ otherwise
  - $L = 2\pi$, $h = 1.5L/k$, $r_1 = 0.3L/k$, $r_2 = 0.4L/k$

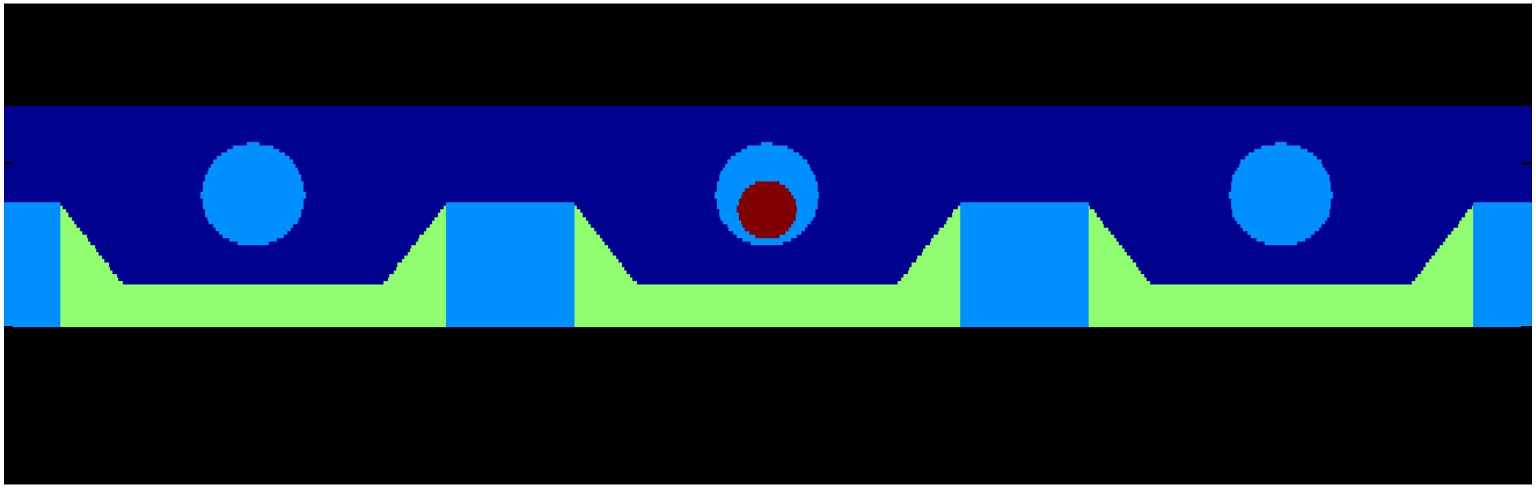


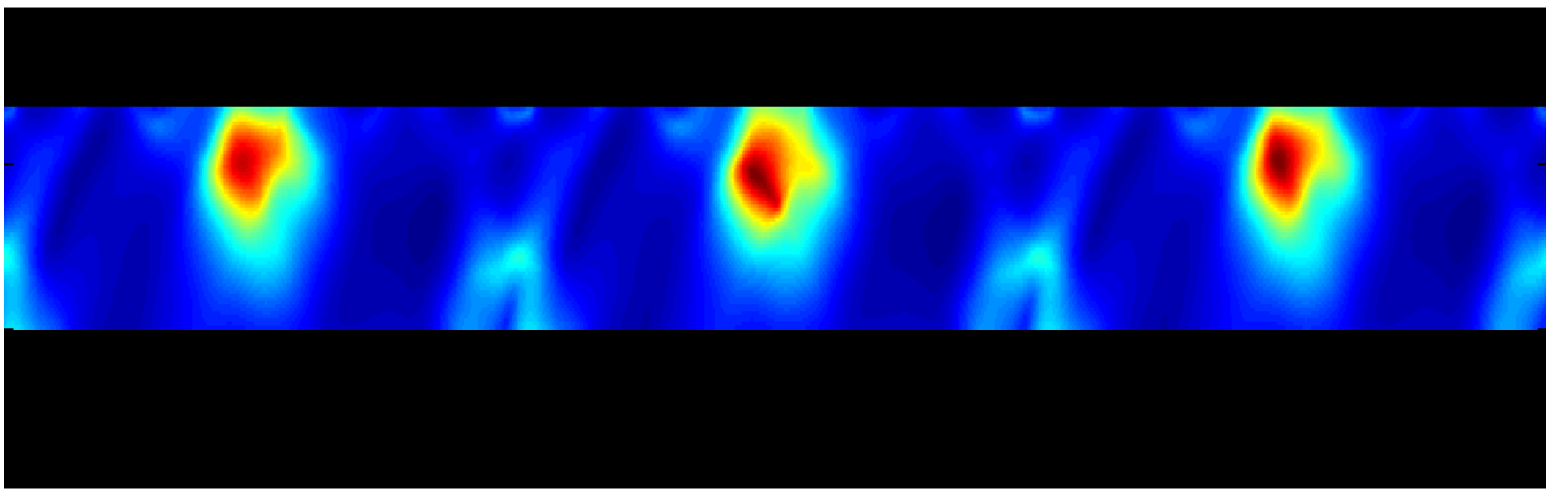
Exact geometry



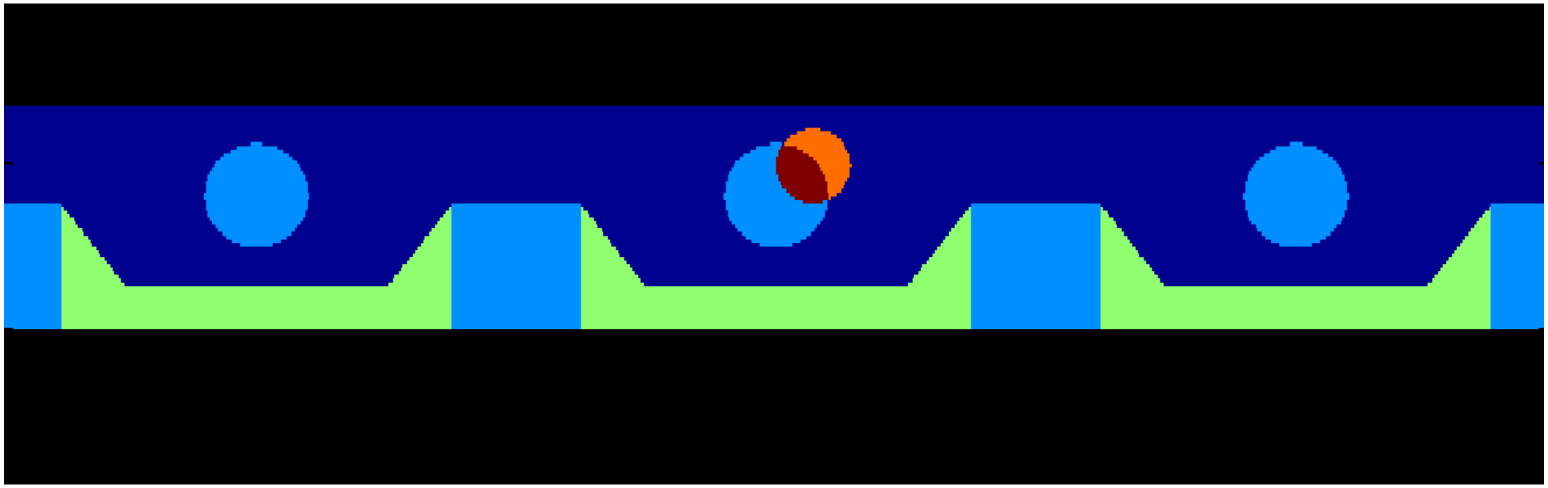
Indicator function $z \mapsto \mathcal{I}_\alpha(z)$

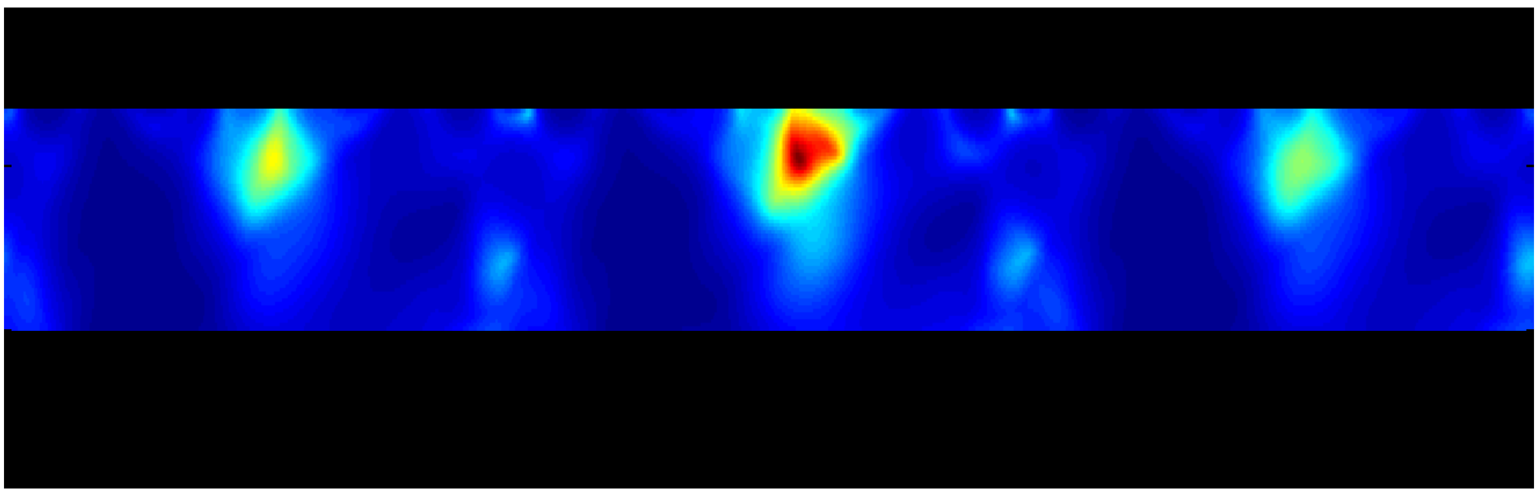
Some numerical results





Some numerical results





Where to go next

With T. Arens we are looking at the scattering by an almost periodic anisotropic layer.

$$\nabla \cdot A \nabla u^s + k^2 n u^s = f \text{ for } (\tilde{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$$

u^s satisfies Rayleigh radiation condition

$\mathrm{supp}(A)$, $\mathrm{supp}(n)$ and $\mathrm{supp}(f)$ are included in $\Omega_h := \mathbb{R}^{d-1} \times (-h, h)$.

$A|_{\Omega_h}$, $n|_{\Omega_h}$ are in the space of Besicovitch-almost periodic bounded function

$$L^\infty_{ap}(\Omega_h) = \overline{P(L^\infty([-h, h])}^{\|\cdot\|_\infty},$$

where for a Banach space X

$$P(X) = \left\{ \sum_{j=1}^{n} u_j \, e^{\lambda_{n_j} \cdot \tilde{x}} : u_j \in X \, , n_j \in \mathbb{N} \, , j = 1, \dots, n \right\}$$

with given countable set of frequencies $\mathcal{S} = \{\lambda_1, \lambda_2, \dots \mid \lambda_j \in \mathbb{R}^{d-1}\}$ which forms a semigroup.

References

- F. Cakoni, H. Haddar, T.P. Nguyen (2019) New interior transmission problem applied to a single Floquet-Bloch mode imaging of local perturbations in periodic media, *Inverse Problems*, **35**, 015009.
- T.P. Nguyen (2019) Differential Imaging of Local Perturbations in Anisotropic Periodic media, *Inverse Problems* (to appear).
- T.Arens, F. Cakoni (2019) Scattering by an almost periodic structure, (preprint).