Inverse scattering by locally rough surfaces with phaseless near-field data

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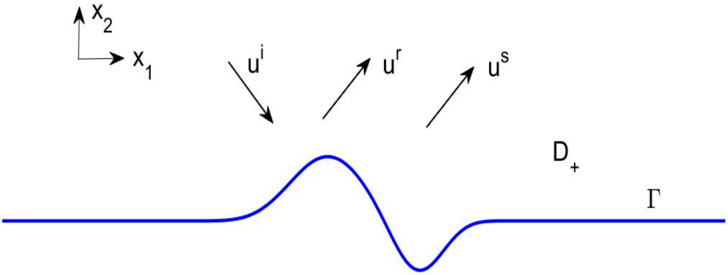
The forward scattering problem

Consider scattering of time-harmonic electromagnetic waves by a locally perturbed, perfectly reflecting, infinite plane (called locally rough surface).

This type of problems occurs in various areas of applications such as radar, sonar, remote sensing and nondestructive testing (in, e.g. materials).



The forward scattering problem

- The scattering problem can be modeled by the Dirichler Problem (DP):

$$\text{Helmholtz equation}: \quad \Delta u^s + k^2 u^s = 0 \quad \text{in } D_+$$

$$\text{Boundary condition}: \quad u^s(x) = -u^i(x) - u^r(x) := f(x) \quad \text{on } \Gamma := \partial D_+$$

$$\text{Radiation condition}: \quad \lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0, \quad r = |x|, \; x \in D_+$$

- Incident wave $u^i(x) = \exp(ikx \cdot d)$
- Reflected wave $u^r(x) = \exp(ikx \cdot d')$ by the infinite plane $x_2 = 0$
- u^s is the unknown scattered field in D_+

The forward scattering problem

The well-posedness of the scattering problem (DP) has been studied:

- Integral Equation Method

Willers, The Helmholtz equation in disturbed half-spaces, Math. Methods Appl. Sci. 9(1987), 312-323.

Zhang-Zhang, A novel integral equation for scattering by locally rough surfaces and application to the inverse problem, SIAM J. Appl. Math. 73(2013), 1811-1829.

- Variational Method

Bao-Lin, Imaging of local surface displacement on an infinite ground plane: the multiple frequency case, SIAM J. Appl. Math. 71(2011), 1733-1752.

The forward scattering problem

By the integral equation given by Zhang-Zhang (2013) we can prove

$$u^s(x,d) = \frac{e^{ik|x|}}{|x|^{1/2}} u^\infty(\hat{x}, d) + u^s_{Res}(x,d) \quad \text{as } |x| \to \infty \tag{1}$$

with

$$\|u^\infty(\cdot, d)\|_{C^1(\mathbb{S}^1_+)} \le C, \tag{2}$$

$$|u^s_{Res}(x,d)| \le \frac{C}{|x|^{3/2}}. \tag{3}$$

- $u^\infty(\hat{x}, d)$: the far-field pattern of the scattering solution u^s

Inverse Problem (with Phaseless Near-Field Data):
Given incident wave u^i and the phaseless total field $|u|^2$ on a surface, determine the locally rough surface Γ

Inverse Scattering with Phaseless Near-Field Data (for bounded scatterers) is also called Phase Retrieval Problems in Optics and has been extensively studied numerically in the past decades:

Maleki-Devaney, Phase-retrieval and intensity-only reconstruction algorithms for optical diffraction tomography, J. Opt. Soc. Amer. A10 (1993), 1086-1092.

Pan-Zhong-Chen-Yeo, Subspace-based optimization method for inverse scattering problems utilizing phaseless data, IEEE T. Geosci. Remot. Sensing 49 (2011), 981-987.

Candes-Li-Soltanolkotabi, Phase retrieval via Wirtinger flow: Theory and algorithms, IEEE T. Inform. Theory 61 (2015), 1985-2007.

Chen-Huang, A direct imaging method for electromagnetic scattering data without phase information, SIAM J. Imag. Sci. 9 (2016), 1273-1297.

Wei-Chen-Qiu-Chen, Conjugate gradient method for phase retrieval based on Wirtinger derivative, J. Opt. Soc. Amer. A34 (2017), 708-712.

X. Chen, Computational Methods for Electromagnetic Inverse Scattering, Wiley, 2018.

Maretzke-Hohage, Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging, SIAM J. Appl. Math. 77 (2017), 384-408 (Stability).

For inverse potential scattering with phaseless near-field data:

$$\Delta u + k^2 u - q(x)u = -\delta(x-y), \ \ x \in \mathbb{R}^3, \ \ x \neq y,$$

- $q \geq 0, \ \ q \in C^2(\mathbb{R}^3)$
- $u = u^i + u^s$ with incident point source $u^i(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$

Klibanov proved the uniqueness results for smooth q:[1]
- q is uniquely determined by phaseless near-field data $|u(x,y,k)|$ or $|u^s(x,y,k)|$, $\forall y \in S$, $\forall x \in B_\varepsilon(y)$, $x \neq y$, $\forall k \in (k_-, k_+)$

Novikov proved the uniqueness result without smoothness on q:[2]
- $q \in L^\infty(\mathbb{R}^3)$ is uniquely determined by phaseless total near-field data $|u(x,y,k)|$, $\forall x, y \in B_{R'} \setminus \overline{B_R}$, fixed k

[1] M.V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.* **74** (2014), 392-410.

[2] R.G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bull. Sci. Math.* **139** (2015), 923-936.

For inverse medium scattering with phaseless near-field data:

$$\Delta u + k^2 n u = -\delta(x-y), \quad x \in \mathbb{R}^3, \quad x \neq y$$

- $n \geq 1, \ n \in C^{15}(\mathbb{R}^3), \ u = u^i + u^s$ with incident point source u^i

Klibanov proved the uniqueness result:[3]

- n is uniquely determined by phaseless near-field data $|u(x,y,k)|$ or $|u^s(x,y,k)|, \forall y \in S, \forall x \in B_\varepsilon(y), \ x \neq y, \forall k \in (k_-, k_+)$

Klibanov-Romanov improved the above uniqueness result:[4]

- n is uniquely determined by phaseless scattered near-field data $|u^s(x,y,k)|, \forall y, x \in S, \ x \neq y, \ \forall k \in (k_-, k_+)$

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[3] M. Klibanov, A phaseless inverse scattering problem for the 3-D Helmholtz equation, *Inverse Probl. Imaging* **11** (2017), 263-276.

[4] Klibanov & Romanov, Uniqueness of a 3-D coefficient inverse scattering problem without the phase information, *Inverse Probl.* **33** (2017) 095007.

Inverse scattering with phaseless far-field data $|u^\infty(\widehat{x}, d)|$

Inverse Scattering with Phaseless Far-Field Data: very few results are available!

Main Difficulty

Translation Invariance Property:
Phaseless far-field pattern is invariant under translations of the obstacle D if only one incident plane wave is used:

$$u_\ell^\infty(\hat{x}; d, k) = e^{ik\ell\cdot(d-\hat{x})} u^\infty(\hat{x}; d, k), \quad \hat{x} \in \mathbb{S}^2, \ \forall \ell \in \mathbb{R}^3$$
$$(\text{or} \ \ |u_\ell^\infty(\hat{x}; d, k)| = |u^\infty(\hat{x}; d, k)| \ \ )$$

for any $\ell \in \mathbb{R}^3$ (Kress-Rundell '97, Liu-Seo '04)

Inverse scattering with phaseless far-field data $|u^\infty(\widehat{x}, d)|$

Only the shape but not the location may be reconstructed from phaseless far-field data:

Kress-Rundell '97, Ivanyshyn '07, Ivanyshyn-Kress '10, '11 (Shape Reconstruction)

Bao-Li-Lv '13 (Perfectly reflecting periodic surfaces, phaseless near-field)

Bao-Zhang '16 (Perfectly reflecting rough surfaces, phaseless near-field)

Li-Liu '15, Li-Liu-Wang '17 (Recovering a polyhedral obstacle by a few backscattering measurements)

Shin '16 (Reconstructing strictly convex sound-soft obstacle by phaseless backscattering data at fixed $k \gg 1$)

Uniqueness for shape reconstruction from phaseless far-field data:

A. Majda, High frequency asymptotics for the scattering matrix and the inverse problem of acoustical scattering, *Comm. Pure Appl. Math.* **29**(1976),261-291: general convex obstacles at high k

X. Liu and B. Zhang, Unique determination of a sound-soft ball by the modulus of a single far field datum, *J. Math. Anal. Appl.* **365** (2010), 619-624: shape of sound-soft disks or balls by one phaseless far-field datum

Stability for shape reconstruction from phaseless far-field data:

H. Ammari, Y. Chow and J. Zou, Phased and phaseless domain reconstructions in the invere scattering problem via scattering coefficients, *SIAM J. Appl. Math.* **76**(2016), 1000-1030: Stability for reconstruction of a small perturbation of a circle from phaseless far-field data

Inverse scattering with phaseless far-field data $|u^\infty(\widehat{x}, d)|$

Progress has been made on inverse scattering with phaseless far-field data:

Translation Invariance Property can be broken by using a superposition of two plane waves as the incident field:

$$u^i = u^i(x; d_1, d_2, k) := \exp(ikd_1 \cdot x) + \exp(ikd_2 \cdot x), \quad d_1 \neq d_2$$

B. Zhang & H. Zhang, Recovering scattering obstacles by multi-frequency phaseless far-field data, *J Comput Phys* **345**(2017), 58-73: Recursive Newton-type iteration method in frequencies

B. Zhang & H. Zhang, Fast imaging of scattering obstacles from phaseless far-field measurements at a fixed frequency, *Inverse Problems* **34**(2018) 104005: Direct imaging method

B. Zhang & H. Zhang, Imaging of locally rough surfaces from intensity-only far-field or near-field data, *Inverse Problems* **33**(2017) 055001: Recursive iteration method for inverse scattering by local rough surfaces with phaseless near-field and far-field data

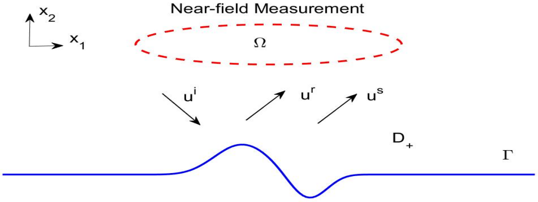
X. Xu, B. Zhang & H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency, *SIAM J. Appl. Math.* **78**(2018), 1737-1753: Under the assumption that the property of the scatterers is a priori known

X. Xu, B. Zhang & H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency. II, *SIAM J. Appl. Math.* **78**(2018), 3024-3039: Removing the a priori assumption on the property of the scatterers by adding a known reference ball to the scattering system

D. Zhang & Y. Guo, Uniqueness results on phaseless inverse scattering with a reference ball, *Inverse Problems* **34**(2018) 085002: Using a superposition of a plane wave and a point source as the incident field and adding a known reference ball to the scattering system

X. Ji, X. Liu & B. Zhang, Target reconstruction with a reference point scatterer using phaseless far field patterns, *SIAM J. Imaging Sci.* **12** (2019), 372-391: Stability and direct imaging method: using one plane wave as the incident field and adding a known point scatterer to the

Inverse rough surface scattering with phaseless near-field data



Theorem (Xu-Z.-Zhang (2019))

[a] *Suppose* Γ_1 *and* Γ_2 *are two locally rough surfaces and* $u_1(\hat{x}, d)$ *and* $u_2(\hat{x}, d)$ *are the total field corresponding to* Γ_1 *and* Γ_2*, respectively. Let* Ω *be a bounded open domain above* Γ_1 *and* Γ_2*. If* $|u_1(x, d_n)| = |u_2(x, d_n)|$ *for all* $x \in \Omega$ *and the distinct directions* $d_n \in \mathbb{S}^1_-$ *with* $n \in \mathbb{N}$ *and a fixed wave number* k*, then* $\Gamma_1 = \Gamma_2$

[a] X. Xu, B. Zhang & H. Zhang, Uniqueness and direct imaging method for inverse scattering by locally rough surfaces with phaseless near-field data, *SIAM J. Imaging Sci.* **12** (2019), 119-152

Main Idea of Proof: Our proof is motivated by the paper

R.G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bull. Sci. Math.* **139**(2015), 923-936.

Step 1. Fix $d = d_n$ for an arbitrary $n \in \mathbb{N}$ and set $d = (d_1, d_2)$. Since $|u_1(x, d)| = |u_2(x, d)|$ for all $x \in \Omega$, we have

$$|u_1(x, d)| = |u_2(x, d)| \quad \text{for} \;\; x \in \mathbb{R}^2_+ \backslash \overline{B_R}. \tag{4}$$

Step 2. $u_l = u^i + u^r + u^s_l$, $l = 1, 2$, and

$$u^s_l(x, d) = \frac{e^{ik|x|}}{|x|^{1/2}} u^\infty_l(\hat{x}, d) + u^s_{l,Res}(x, d), \quad l = 1, 2 \tag{5}$$

with

$$|u^s_{l,Res}(x, d)| \le C|x|^{-3/2}, \qquad |u^s_l(x, d)| \le C|x|^{-1/2} \tag{6}$$

for $x \in D_{+,l}$ with $|x|$ large enough.

Step 3. Write

$$u_l^\infty(\hat{x}, d) = r_l(\hat{x}, d)e^{i\theta_l(\hat{x},d)}, \quad l = 1, 2,$$

where $r_l \geq 0$ and $\theta_l \in [0, 2\pi]$. Then, by (4) we have

$$\begin{aligned} & r_1(\hat{x}, d)\sin(\alpha|x|)\sin[\theta_1(\hat{x}, d) + \beta|x|] + \frac{1}{2}v_1(x, d) \\ &= r_2(\hat{x}, d)\sin(\alpha|x|)\sin[\theta_2(\hat{x}, d) + \beta|x|] + \frac{1}{2}v_2(x, d), \end{aligned} \tag{7}$$

where $\alpha = k\hat{x}_2 d_2 < 0$, $\beta = k(1 - \hat{x}_1 d_1) > 0$ and

$$|v_l(x, d)| \leq \frac{C}{|x|^{1/2}} \quad \text{as} \quad |x| \to +\infty, \quad l = 1, 2. \tag{8}$$

Step 4. Choose $\gamma_0^{(1)}, \gamma_0^{(2)} \in \mathbb{R}$ such that

$$\sin\left(\frac{\alpha}{\beta}\gamma_0^{(k)}\right) \neq 0, \quad k = 1, 2, \tag{9}$$

$$\sin(\gamma_0^{(1)} - \gamma_0^{(2)}) \neq 0. \tag{10}$$

We aim to prove that

$$r_1 \sin(\theta_1 + \gamma_0^{(k)}) = r_2 \sin(\theta_2 + \gamma_0^{(k)}), \quad k = 1, 2, \tag{11}$$

where we write $r_l = r_l(\hat{x}, d)$, $\theta_l = \theta_l(\hat{x}, d)$, $l = 1, 2$, for simplicity.

Case 1. α/β is a rational number. There exist $p_j \in \mathbb{N}$ with $j = 1, 2, \ldots$ such that $(\alpha/\beta)p_j \in \mathbb{N}$ and $\lim_{j\to+\infty} p_j = +\infty$.

Let $x_j^{(k)} := (\gamma_0^{(k)} + 2\pi p_j)\hat{x}/\beta$, $k = 1, 2$. Then $x_j^{(k)} \in \mathbb{R}_+^2 \backslash \overline{B}_R$ for large j and $\lim_{j\to+\infty} |x_j^{(k)}| = +\infty$. Taking $x = x_j^{(k)}$ in (7) and then letting $j \to +\infty$ give (11).

Case 2. α/β is an irrational number. By Kronecker's approximation theorem, $\exists\ p_j \in \mathbb{N}$, $j = 1, 2, \ldots$, such that $(\alpha/\beta)p_j = m_j + a_j$ with $m_j \in \mathbb{N}$, $\lim_{j\to+\infty} a_j = 0$ and $\lim_{j\to+\infty} p_j = +\infty$.
Taking $x = x_j^{(k)} := (\gamma_0^{(k)} + 2\pi p_j)\hat{x}/\beta$ in (7) and letting $j \to +\infty$ give (11).

Step 5. From (11) it follows that

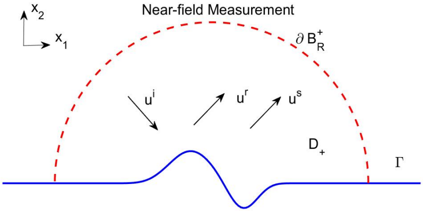
$$\begin{pmatrix} \cos\gamma_0^{(1)} & \sin\gamma_0^{(1)} \\ \cos\gamma_0^{(2)} & \sin\gamma_0^{(2)} \end{pmatrix} \begin{pmatrix} r_1\sin\theta_1 - r_2\sin\theta_2 \\ r_1\cos\theta_1 - r_2\cos\theta_2 \end{pmatrix} = 0$$

Condition (10) means that $r_1\sin\theta_1 = r_2\sin\theta_2$, $r_1\cos\theta_1 = r_2\cos\theta_2$, so $u_1^\infty = u_2^\infty$ or

$$u_1^\infty(\hat{x}, d_n) = u_2^\infty(\hat{x}, d_n), \ \ \forall\hat{x} \in \mathbb{S}_+^1, \ \ d_n \in \mathbb{S}_-^1, \ \ n \in \mathbb{N}$$

Based on uniqueness result for inverse rough surface scattering with full far-field data (Zhang-Zhang, 2013), we obtain that $\Gamma_1 = \Gamma_2$.

Inverse rough surface scattering with phaseless near-field data: Direct imaging method



Consider the imaging function

$$I^{Phaseless}(z) := \int_{\partial B_R^+} \left| \int_{\mathbb{S}^1_-} \left[\left(|u(x,d)|^2 - 2 + e^{2ikx_2 d_2} \right) e^{ik(x-z)\cdot d} - e^{ik(x'-z')\cdot d} \right] ds(d) \right|^2 dx$$

for $z \in \mathbb{R}^2$.

For $z \in \mathbb{R}^2$ define the function

$$F(R,z) := \int_{\partial B_R^+} |U(x,z)|^2 \, dx, \qquad (12)$$

where

$$\begin{aligned} U(x,z) \quad := \quad & \int_{\mathbb{S}^1_-} u^s(x,d) e^{-ikz\cdot d} ds(d) - \int_{\mathbb{S}^1_-} e^{ik(x\cdot d' - z\cdot d)} ds(d) \\ & - \int_{\mathbb{S}^1_-} e^{ik(x\cdot d' - z'\cdot d)} ds(d) \end{aligned}$$

Direct imaging method

Lemma

For $z \in \mathbb{R}^2$ *and* $R > 0$ *we have* $F(R,z) = F_0(z) + F_{0,Res}(R,z)$, *where*

$$F_0(z) := \int_{\mathbb{S}^1_+} \left| \int_{\mathbb{S}^1_-} u^\infty(\hat{x},d) e^{-ikz\cdot d} ds(d) - \left(\frac{2\pi}{k}\right)^{1/2} e^{-\frac{\pi}{4}i} \left(e^{-ik\hat{x}\cdot z'} + e^{-ik\hat{x}\cdot z} \right) \right|^2 ds(\hat{x})$$

which is independent on R, *and* $F_{0,Res}(R,z)$ *satisfies the estimate*

$$|F_{0,Res}(R,z)| \leq C \frac{(1+|z|)^4}{R^{1/4}} \tag{13}$$

for sufficiently large R. *Here,* $C > 0$ *is a constant independent of* R *and* z.

Proof is based on the method of stationary phase with error bounds:

F.W.J. Olver, Error bounds for stationary phase approximations, *SIAM J. Math. Anal.* **5** (1974), 19-29.

Direct imaging method

Theorem

For $z \in \mathbb{R}^2$ and $R > 0$ we have

$$I^{Phaseless}(z) = F(R,z) + F_{Res}(R,z), \tag{14}$$

where $F(R,z)$ is defined in (12) and $F_{Res}(R,z)$ satisfies the estimate

$$|F_{Res}(R,z)| \leq C\frac{(1+|z|)^2}{R^{1/3}} \tag{15}$$

for R large enough and $C > 0$ independent of R and z.

$$\begin{aligned} I^{Phaseless}(z) &= F_0(z) + F_{0,Res}(R,z) + F_{Res}(R,z) \\ &\approx F_0(z) =: I^{Full}(z) \end{aligned}$$

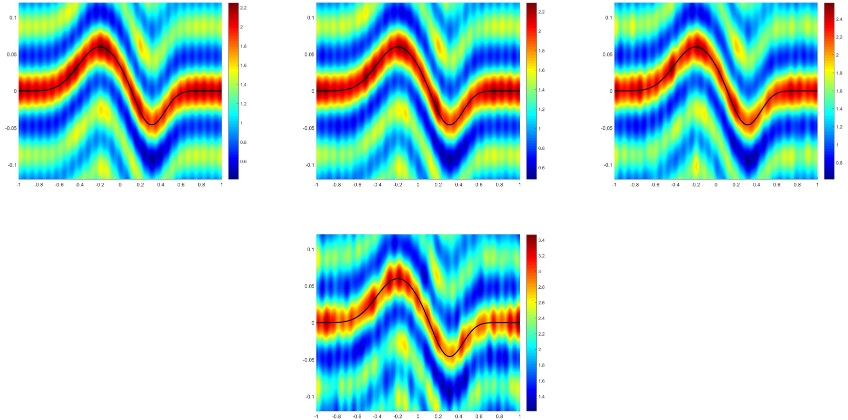


Figure: Imaging results of $I^{Phaseless}(z)$ with no noise, 10% noise, 20% noise and 40% noise, respectively, where $k = 40$, $R = 4$

Numerical experiments

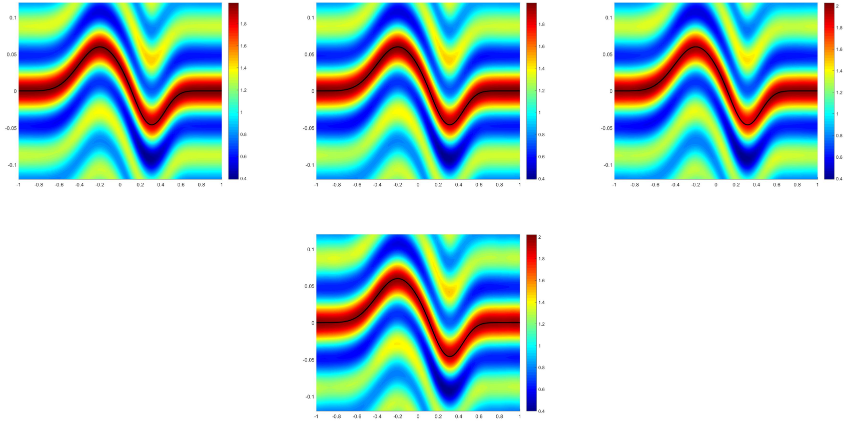


Figure: Imaging results of $I^{Full}(z)$ with no noise, 10% noise, 20% noise and 40% noise, respectively, where $k = 40$

Numerical experiments

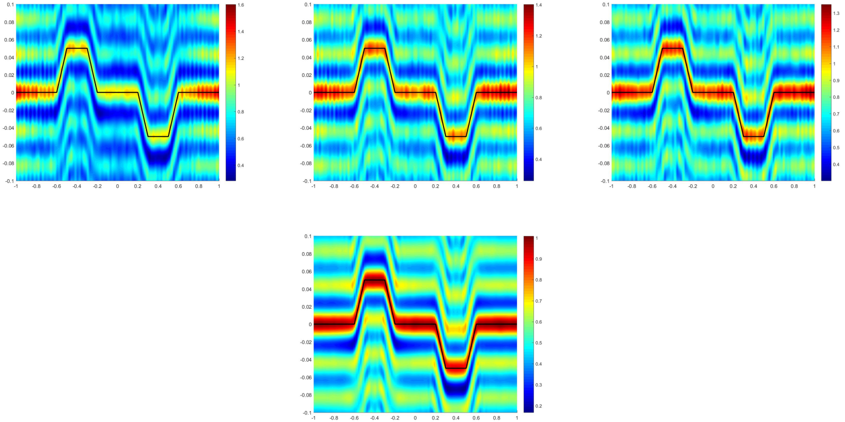


Figure: Imaging results of $I^{Phaseless}(z)$ with 20% noise and R=1.2, R=1.6 and R=2, respectively, and of $I^{Full}(z)$, where $k = 80$

Numerical experiments

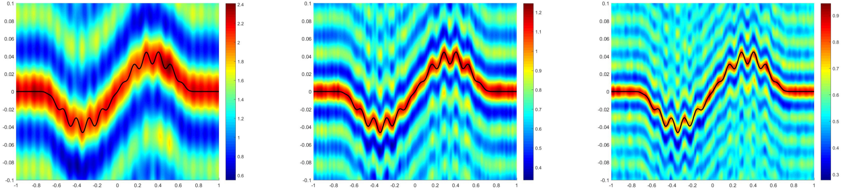


Figure: Imaging results of $I^{Phaseless}(z)$ with 20% noise and k=40, k=80 and k=120, respectively, where $R = 4$

Numerical experiments

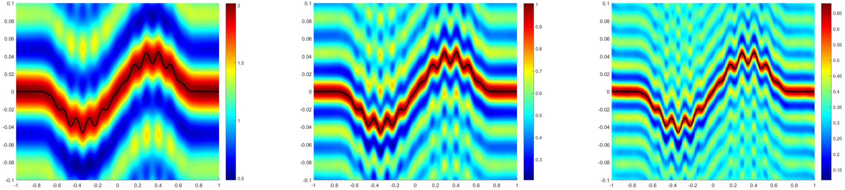


Figure: Imaging results of $I^{Full}(z)$ with 20% noise and k=40, k=80 and k=120, respectively.

Thank You!