Uncertainty and Innovation*

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Abstract

Investors must typically decide whether or not to fund an innovative project with very limited knowledge of the odds of success, a situation that is best described as "Knightian uncertainty." This paper argues that innovation waves can be the product of investors' uncertainty (or "ambiguity") aversion. We show that uncertainty-averse investors are more willing to accept an uncertain lottery, such as investing in a new innovative venture, if they can also take other uncertain lotteries, that is, to make contemporaneous investments in other innovative ventures. This means that uncertainty aversion makes investment in innovative projects strategic complements, which results in innovation waves. We also show that innovation waves may be sparked by favorable technological shocks in one sector, and then spill over to other contiguous sectors. Thus, innovation waves ripple through the economy. We also argue that uncertainty aversion has implications for the composition of venture capital portfolios, and the structure of the venture capital industry.

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Innovation is the most important driver of economic wealth in the modern world. There are times when innovation is stagnant, but other times when technology leaps forward. Furthermore, investors must typically decide whether or not to fund an innovative project with very limited knowledge of the odds of success, a situation that is best described as "Knightian uncertainty" (Knight, 1921). In this paper, we study the impact of uncertainty aversion on the incentives to innovate and we show that uncertainty aversion can generate innovation waves associated with stock market booms.

There are many reasons why innovation develops in waves. These include fundamental reasons such as random technological breakthroughs in the presence of network externalities. In this paper, we focus on the incentives to create and finance innovations. We argue that innovation waves can be the product of investors' uncertainty aversion. We show that investors' uncertainty aversion creates externalities in innovative activities which may result in innovation waves. We also show that innovation waves are associated with stock market booms in technology sectors.

We study an economy with multiple entrepreneurs endowed with project-ideas. Project-ideas are risky and, if successful, may lead to innovations. These project-ideas are potentially competing, for example because it is unclear which of alternative new innovative products will be most preferred by customers.[1] The innovation process consists of two stages. In the first stage, entrepreneurs must decide whether or not to invest personal resources, such as effort, to innovate. If the first stage of the process is successful, further development of the innovation requires additional investment in the second stage. Entrepreneurs raise funds for the additional investment by selling shares of their firms to uncertainty-averse investors. The second stage of the innovation process is uncertain in that outside investors are uncertain of the exact distribution of the residual success

[1] For example, competition between Apple and Samsung in the smart phone market is surrounded by considerable uncertainty on the product features that are most valued by customers.

probability of the innovation process. Following Epstein and Schnieder (2011), we model uncertainty aversion by assuming that outside investors are Minimum Expected Utility (MEU) maximizers and that they hold a set of priors, or "beliefs," rather than a single prior as it is the case for Subjective Expected Utility (SEU) agents.

We show that uncertainty-averse investors prefer to hold an uncertain asset only if they can also hold other uncertain assets in their portfolios, a feature denoted as uncertainty hedging. Because of uncertainty hedging, an investor will be more "optimistic" on an innovation if he/she is able to invest in other innovations as well. Thus, investors are willing to pay more for equity in a given entrepreneur's firm when other entrepreneurs innovate and issue equity as well. This means that investors are more willing to fund an entrepreneur's innovation if they can also fund other entrepreneurs at the same time. It also means that the market value of equity of a firm will be greater when multiple firms are on the market as well. Thus, investments in different innovative companies are effectively complements.

We show that investors' uncertainty aversion generates inefficient equilibria where potentially valuable innovation is not pursued. When the initial personal costs to the entrepreneur is sufficiently low, entrepreneurs' dominant strategy is to innovate, irrespective of other entrepreneurs' decisions. Similarly, when the initial personal cost is very large, the dominant strategy is not to innovate. For intermediate levels of the initial personal cost, an entrepreneur is willing to initiate the innovation process only if she expect also other entrepreneurs to innovate as well. Thus, multiple equilibria, with and without innovation, may exists. Existence of the inefficient equilibrium without innovation depends on the correlation between the success rates of the innovation processes, that is on the degree of "relatedness" of the innovation.

Strategic complementarity between innovative activities due to uncertainty aversion may results in innovation waves. An innovation wave may be sparked by favorable technological shocks in one sector that trigger an entrepreneur to initiate an innovation. Because of the innovation in one company, other entrepreneurs expect now more favorable pricing of their equity by investors, inducing them to innovate as well. Thus innovation in one firm can then spill over to other firms even in absence of explicit technological spillover between the two firms. In this way, innovation waves ripple through the economy.

Complementarity between innovation activities due to uncertainty aversion may also results in technology sector equity market booms. To see this note that the link between the innovation activities of firms is due to the positive effect of the innovation of one firm on the equity pricing of other firms. This means that innovation in one sector may trigger a positive effect on equity valuations that spills to contiguous sectors. Thus, "equity market booms" in technology markets can materialize. These booms are beneficial since they can spur valuable innovation.

We also argue that uncertainty aversion has implications for the composition of venture capital portfolios, and the structure of the venture capital industry. This happens because of the possible beneficial role that venture capitalists can play to remedy a coordination failure that causes the inefficient no-innovation equilibrium.

This paper belongs to the rapidly expanding literature on the determinants of innovation and innovation waves (see Fagerberg, Mowery and Nelson, 2005, for an extensive literature review).[2] The critical role of innovation and innovation waves in modern economies has been extensively studied at least since Schumpeter (1939) and (1942), Kuznets (1940), Kleinknecht (1987) and, more recently, Aghion and Howitt (1992). Early research focused mostly on the "fundamentals" behind innovation waves, such as the positive spillover effects across different technologies. More recent research has focused on the link between inno-

[2] See also Chemmanur and Fulghieri (2014) for a discussion of current issues related to entrepreneurial finance and innovation.

vation waves, the availability of financing and stock market booms. Scharfstein and Stein (1990) suggest that reputation consideration by investment managers may induce herd their behavior in the stock market, and thus facilitate the financing of technology firm. Gompers and Lerner (2000) find that higher venture capital valuations are not necessarily linked to better success rates of portfolio companies. Perez (2002) shows that technological revolutions are associated with "overheated" financial markets. Gompers et al. (2008) suggest that increased venture capital funding is the rational response to positive signals on technology firms' investment opportunities. Nanda and Rhodes-Kropf (2013) find that in "hot markets" VCs invest in riskier and more innovative firms. Nanda and Rhodes-Kropf (2014) argue that favorable financial market conditions reduce refinancing risk for VCs, promoting investment in more innovative projects. In a closely related paper, Garlappi, Giammarino, and Lazrak (2014) examine the impact of uncertainty aversion on firm governance, investment and the choice of financing.

To our knowledge, our is the first paper that models explicitly the role of uncertainty aversion on the innovation process and its impact on innovation waves and stock market valuations. We show that investors' uncertainty aversion can generate innovation waves that are driven by investors' beliefs. In our model, due to uncertainty aversion, investors' beliefs are endogenous, and they respond to the availability of investments in innovative projects. Innovation waves and stock market "exuberance" are jointly determined in equilibrium in a model where investors are rational. Greater investment in innovation is combined with greater investors' optimism.

The paper is organized as follows. In Section 1, we briefly discuss the model of uncertainty aversion that is at the foundation of our analysis. In Section 2, we introduce the basic model. In Section 3, we derive the paper's main results. Section 4 presents the main empirical implications of our model. Section 5 concludes. All proofs are in the Appendix.

1 Uncertainty Aversion

A common feature of current economic models is to assume that all agents know the distribution of all possible outcomes.[3] An implication of this assumption is that there is no distinction between the known-unknown and the unknown-unknown. However, the Ellsberg paradox shows that this implication is not warranted.[4] This introductory section briefly describes how various models have accounted for risk and uncertainty.

In traditional models, economic agents maximize their Subjective Expected Utility (SEU). Given a von-Neumann Morgenstern (vNM) utility function u and a probability distribution over wealth, μ, each player maximizes

$$U^e = E_\mu \left[u \left(w \right) \right]. \tag{1}$$

One limitation of the SEU approach is that it cannot account for aversion to uncertainty, or "ambiguity." In the SEU framework, economic agents merely average over the possible probabilities. Under uncertainty aversion, a player does not know the true prior, but only knows that the prior is from a given set, $\mathcal{M}$.

A common way for modeling uncertainty (or ambiguity) aversion is the minimum expected utility approach (MEU), promoted in Epstein and Schneider (2011). In this frame-

[3] This section draws on Dicks and Fulghieri (2014a).

[4] A good illustration of the Ellsberg paradox is actually from Keynes (1921). There are two urns. Urn K has 50 red balls and 50 blue balls. Urn U has 100 balls, but the subject is not told how many of them are red (all balls are either red or blue). The subject will be given \$100 if the color of their choice is drawn, and the subject can choose which Urn is drawn from. Subjects typically prefer urn K, revealing aversion to ambiguity (this preference is shown to be strict if the subject receives \$101 from selecting Urn U but \$100 from Urn K being drawn). To see this, suppose the subject believes that the probability of drawing Blue from Urn U is p. If p $< \frac{1}{2}$, the subject prefers to draw Red from Urn U. If p $> \frac{1}{2}$, the subject prefers to draw Blue from Urn U. If p $= \frac{1}{2}$, the subject is indifferent. Because subjects strictly prefer to draw from Urn K, such behavior cannot be consistent with a single prior on Urn U. This paradox provides the motivation for the use of multiple priors. Further, the subject's beliefs motivate the failure of additivity of asset prices: in this example, the subject believes that $p_B + p_R < p_{(B \cup R)} = 1$.

works, economic agents maximize

$$U^a = \min_{\mu \in \mathcal{M}} E_\mu \left[u\left(w\right) \right]. \tag{2}$$

As shown in Gilboa and Schmeidler (1989), the MEU approach is a consequence of replacing the Sure-Thing Principle of Anscombe and Aumann (1963), with the Uncertainty Aversion Axiom.[5] This assumption captures the intuition that economic agents prefer risk to uncertainty – they prefer known probabilities to unknown. MEU has the intuitive feature that a player first calculates expected utility with respect to each prior, and then takes the worst-case scenario over all possible priors. In other words, the agent follows the maxim "Average over what you know, then worry about what you don't know."[6]

In this paper, we use the MEU approach with recursively defined utilities, as described in Epstein and Schnieder (2011). Formally, we will model *sophisticated uncertainty-averse* economic agents with consistent planning. In this setting agents are sophisticated in that they correctly anticipate their future uncertainty aversion. Consistent planning accounts for the fact that agents take into account how they will actually behave in the future.[7] Our results are smooth (a.e.) because we explore a setting where we can apply a minimax theorem.

An important feature of uncertainty aversion that will play a critical role in our paper is that agents may benefit from diversification, a feature that we will refer to as *uncertainty hedging*. This feature can be seen as follows. Consider two random variables, y_k, $k \in \{1, 2\}$, with distribution $\mu_k \in \mathcal{M}$, which is ambiguous to agents. Uncertainty-hedging is the

[5] Anscombe and Aumann (1963) is an extention of the Savage (1972) framework: the Anscombe and Aumann framework has both objective and subjective probabilities, while the Savage framework has only subjective probabilities.

[6] Another approach is the smooth ambiguity model developed by Klibanoff, Marinacci, and Mukerji (2005). In their model, agents maximize expected felicity of expected utility. Agents are ambiguity averse if the felicity function is concave.

[7] Siniscalchi (2011) describes this framework as preferences over trees.

property that uncertainty-averse agents prefer to pick the worst-case scenario for a portfolio, rather than choosing the worst-case scenario for each individual asset in its portfolio.[8]

Theorem 1 *Ambiguity-averse agents prefer uncertainty-hedging:*

$$\begin{aligned} q \min_{\mu\in\mathcal{M}} E_\mu\left[u\left(y_1\right)\right] + (1-q) \min_{\mu\in\mathcal{M}} E_\mu\left[u\left(y_2\right)\right] &\leq \\ \min_{\mu\in\mathcal{M}} \{qE_\mu\left[u\left(y_1\right)\right] + (1-q)E_\mu\left[u\left(y_2\right)\right]\}, \quad \textit{for all } q &\in [0,1]. \end{aligned} \tag{3}$$

If agents are SEU, (3) holds as an equality.

This property will play a key role in our model. It implies that uncertainty-averse agents prefer to hold a portfolio of uncertain assets rather than a single uncertain asset, because investors can lower their exposure to uncertainty by holding a diversified portfolio. Alternatively, it suggests that an investor will be more "optimistic" about a portfolio than about a single asset. Thus, uncertainty hedging creates a complementarity between assets for investors so the value investors place on a given asset is increasing in their portfolio exposure to other assets.[9]

A second critical feature of our model is that we do not impose rectangularity of beliefs (as in Epstein and Schneider 2003). Rectangularity of beliefs effectively implies that prior beliefs in the set of admissible priors can be chosen independently from each other.[10] In our model, we assume that the agent faces a restriction on the set of the core beliefs $\mathcal{M}$

[8] Note that, as such, property (3) is reminiscent of the well-known feature that a portfolio of options is worth more than an option on a portfolio and, thus, that writing a portfolio of options is more costly than writing an option on a portfolio.

[9] We will show that such portfolio complementarity will induce entrepreneurs to exhibit strategic complementarity in their innovation decisions, resulting in multiple equilibria, because an entrepreneur is more willing to innovate if they expect other entrepreneurs are innovating as well. Dicks and Fulghieri (2014b) shows that uncertainty hedging also causes systemic risk, in that idiosyncratic shocks spread into financial crises.

[10] Rectangularity of beliefs is commonly assumed to guarantee dynamic consistency. However, Aryal and Stauber (2014) show that, with multiple players, rectangularity of beliefs is not sufficient for dynamic consistency.

over which the minimization problem (2) is taking place. These restrictions are justified by the observation that the nature of the economic problem imposes certain consistency requirements in the set of the core beliefs $\mathcal{M}$. In other words, we recognize that the "fundamentals" of the economic problem faced by the uncertainty-averse agent generates a loss of degree of freedom in the selection of prior beliefs.[11]

2 The Basic Model

We study a two-period model, with three dates, $t \in \{0, 1, 2\}$. The economy has two classes of agents: investors and (two) entrepreneurs. Entrepreneurs are endowed with unique project-ideas that may lead to an innovation. Project-ideas are risky and require investments both at the beginning of the period, $t = 0$, and at the interim date, $t = 1$, as discussed below; if successful, project-ideas generate an innovation at the end of the second period, $t = 2$. If the project-idea is unsuccessful, it will have zero payoff. For simplicity, we assume initially that there are only two types of project-ideas, denominated by τ, with $\tau \in \{A, B\}$.

Entrepreneurs are penniless and require financing from investors. There is a continuum of investors (of measure one). Investors are endowed at the beginning of the first period, $t = 0$, with w_0 units of the riskless asset. The riskless asset can either be invested in one of the two types a project-idea, or it can invested in the riskless technology. A unit investment in the riskless technology, which can be made either at $t = 0$ or $t = 1$, yields a unit return in the second period, $t = 2$, so that the (net) riskless rate of return is zero.

We assume that project-ideas are specific to each entrepreneur, that is, an entrepreneur

[11] For example, an uncertainty-averse producer may face uncertainty on the future consumption demand exerted by her customers. The beliefs held by the uncertainty-averse agent on consumer demand must be consistent with basic restrictions, such as the fact that the consumer choices must satisfy an appropriate budget constraint.

can invest in only one type of project-ideas, which will determine entrepreneur's type τ, $\tau \in \{A, B\}$. This assumption captures the notion that project-ideas are creative innovations that can be successfully pursued only by the entrepreneur who generated them.

The innovation process is structured in two stages. To implement a project-idea, and thus "innovate," an entrepreneur must first pay at $t = 0$ a discovery cost k_τ. The cost k_τ represents all the preliminary personal effort that the entrepreneur must exert in order to generate the idea and make it viable. Let q_τ be the success probability of the first stage of the innovation process. We assume that the first-stage success probabilities of the two project-ideas are correlated. Specifically, we assume that the probability that both entrepreneurs are successful in the first stage is $q_A q_B + r$, while the probability that entrepreneur τ is successful if entrepreneur τ' is not successful is $q_\tau(1 - q_{\tau'}) - r$, with $\tau', \tau \in \{A, B\}$, $\tau' \neq \tau$ and $r \in \left[-\min\left\{q_A q_B, (1 - q_A)(1 - q_B)\right\}, \min_\tau q_\tau (1 - q_\tau)\right]$.[12] Thus, the parameter r capture the notion of possible similarity between entrepreneurial project-ideas, that is, the degree of "relatedness" of their innovations.

If the first stage of the innovation process is successful, at $t = 1$ entrepreneurs enter its second stage. In the second stage, the entrepreneur must decide the level of innovation intensity (for example, the level of R&D expenditures) of the innovation process. Innovation intensity will affect the ultimate value of the innovation, denoted by y_τ. Innovation intensity is costly, and we assume that entrepreneur of type τ sustain a cost $c_\tau(y_\tau) = \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, where Z_τ represents the entrepreneur's productivity. We will assume that the productivity parameters, Z_τ, for the two entrepreneurs are not too dissimilar.[13] If the second stage of the innovation process is successful, which happens with probability p_τ, the innovation process will have a value y_τ at the end of the second period, $t = 2$.

[12] It can be quickly verified that the correlation of the first-stage projects is $r\left[q_A(1 - q_A) q_B (1 - q_B)\right]^{-\frac{1}{2}}$.

[13] Formally, we assume that $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi = \frac{1}{4} e^{2\left(\theta^e - \hat{\theta}_0\right)(\gamma+1)} \left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$. This assumption guarantees that if both first-stage projects are successful, entrepreneurs execute innovation intensity levels so that investors have interior beliefs in equilibrium.

We assume that entrepreneurs are impatient and that will sell at the interim period, $t = 1$, their firms to outside investors at total price V_τ. Investors, however, are uncertain on the success probability of project-ideas. Following Dicks and Fulghieri (2014a), we model uncertainty aversion by assuming the success probability of an innovation of type-τ depends on the value of an underlying parameter θ, and is denoted by $p_\tau(\theta)$. Uncertainty-averse investors treat the parameter θ as uncertain, and believe that $\theta \in C \equiv \left[\hat{\theta}_0, \hat{\theta}_1\right]$, where C represents the set of "*core beliefs*" (similar but less elegant solutions hold for more general $C \subset [\theta_0, \theta_1]$). For analytical tractability, we assume that $p_A(\theta) = e^{\theta - \theta_1}$ and $p_B(\theta) = e^{\theta_0 - \theta}$.[14] In this specification, increasing the value of the parameter θ increases the success probability of type-A project-ideas and decreases the success probability of type-B project-ideas. This means that a greater value of θ is "favorable" for innovation A and "unfavorable" for innovation B. Also, for a given value of the parameter θ, the probabilities distributions $p_\tau(\theta)$, $\tau \in \{A, B\}$, are independent.[15] We will also assume that the core of beliefs is symmetric, so that $\theta_1 - \hat{\theta}_1 = \hat{\theta}_0 - \theta_0$. Finally, we let $\theta^e \equiv \frac{1}{2}(\theta_0 + \theta_1)$. We will at times benchmark the behavior of uncertainty-averse investors with the behavior of uncertainty-neutral, or SEU, investors, and we will assume that uncertainty-neutral investors believe that $\theta = \theta^e$, differently from uncertainty-averse investors who believe that $\theta \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$.

Payoffs are determined as follows. If entrepreneur τ innovates, she produces an innovation with value y_τ and sells the firm for value V_τ. Thus, by holding a risky portfolio $\{\omega_A, \omega_B\}$, an investor will have portfolio $\Pi = \{\omega_A y_A, \omega_B y_B, w_0 - \omega_A V_A - \omega_B V_B\}$. We assume that the budget constraint is nonbinding in equilibrium: $w_0 > \omega_A V_A + \omega_B V_B$. All

[14] This assumption allows us to dispense with rectangularity of beliefs in a tractable way, but is not necessary. Our paper's main results go through for $p_\tau(\theta_\tau)$, with $\theta_\tau \in [\theta_0, \theta_1]$, as long as the core belief set C is a strictly convex, compact subset of $[\theta_0, \theta_1]^2$ with a smooth boundary, such that $\{\theta_A, \theta_B\} \in C$.

[15] Our model can easily be extended to the case where, given θ, the realization of the asset payoffs at the end of the period are correlated.

players are risk-neutral.

2.1 Endogenous Beliefs

An important implication of uncertainty aversion is that the investors' belief on the parameter θ depend on their overall exposure to the source of risk in the economy and, thus, on the structure of their portfolios.[16] If an investor decides to buy ω_A of entrepreneur A's firm, which has produced y_A of type-A innovations, and ω_B of entrepreneur B's firm, which has produced y_B of type-B innovations, she will have portfolio $\Pi = \{\omega_A y_A, \omega_B y_B, w_0 - \omega_A V_A - \omega_B V_B\}$. Because investors are uncertainty averse (they believe $\theta \in C$), portfolio Π provides the investor with utility

$$U(\Pi) = \min_{\theta \in C} \left\{ e^{\theta - \theta_1} \omega_A y_A + e^{\theta_0 - \theta} \omega_B y_B + w_0 - \omega_A V_A - \omega_B V_B \right\}.$$

Because of uncertainty aversion, the investor's belief at $t = 1$ on the state of the economy, θ^a, is given by the solution to the following minimization problem:

$$\theta^a(\Pi) = \arg\min_{\theta \in C} U(\Pi),$$

and is characterized in the following lemma.

Lemma 1 *Let*

$$\tilde{\theta}^a(\Pi) = \theta^e + \frac{1}{2} \ln \frac{\omega_B y_B}{\omega_A y_A}. \tag{4}$$

The belief held by an uncertainty-averse agent with portfolio

[16] For additional discussion, see Dicks and Fulghieri (2014a).

$\Pi = \{\omega_A y_A, \omega_B y_B, w_0 - \omega_A V_A - \omega_B V_B\}$ *is*

$$\theta^a(\Pi) = \begin{cases} \hat{\theta}_0 & \tilde{\theta}^a(\Pi) \leq \hat{\theta}_0 \\ \tilde{\theta}^a(\Pi) & \tilde{\theta}^a(\Pi) \in \left(\hat{\theta}_0, \hat{\theta}_1\right) \\ \hat{\theta}_1 & \tilde{\theta}^a(\Pi) \geq \hat{\theta}_1 \end{cases} . \tag{5}$$

Lemma 1 shows an investor's beliefs depend crucially on his portfolio, Π. We will refer to $\tilde{\theta}^a(\Pi)$ as the "*porfolio-distorted*" beliefs. We will say that the agent has *interior beliefs* when $\theta^a \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$. In this case, the agent's beliefs are equal to the portfolio-distorted beliefs, that is $\theta^a(\Pi) = \tilde{\theta}^a(\Pi)$. Otherwise, we will say that the investor holds "*corner beliefs*." Note that the beliefs of an uncertainty-averse investor depend essentially on the composition of her portfolio Π. The following lemma can be immediately be verified.

Lemma 2 *Holding type-τ innovations constant, an increase in an investor's holdings of type-τ' innovations, $\omega_{\tau'} y_{\tau'}$ with $\tau' \neq \tau$, makes the investor more optimistic about type-τ innovations, for $\tau \in \{A, B\}$. In addition, portfolio-distorted beliefs are homogeneous of degree zero in the holding of the risky innovations, $\{\omega_A y_A, \omega_B y_B\}$. Finally, if only one entrepreneur has a successful first-stage project-idea, the market is more pessimistic about his project than if both entrepreneurs have a successful first-stage project-idea.*

Lemma 2 shows that when a investor has a relatively greater proportion of her portfolio invested in innovation τ (determined, for example, by a decrease in an investor's holding in type-τ'), she will be relatively more concerned about the priors that are less favorable to that innovation. Thus, the investor will give more weight to the states of nature that are less favorable for that innovation. In other words, the investor will be more "pessimistic" about the return on that innovation. Correspondingly, the investor will become more "optimistic" with respect to the other innovation. Proportional changes in an investor's position in the risky innovations will not affect her belief.

Lemma 2 also shows an interesting implication of Lemma 1. Suppose entrepreneur A decides to innovate, but entrepreneur B decides not to. Because $y_B = 0$, by Lemma 1, $\theta^a(\Pi) = \hat{\theta}_0$ for any $\omega_A y_A > 0$. Similarly, if entrepreneur B enters but entrepreneur A does not, $\theta^a(\Pi) = \hat{\theta}_1$.

We will now solve the model recursively. First, we will find the optimal innovation intensity of an entrepreneur, then we will consider the initial decision to innovate.

3 The Innovation Decision

We now characterize the two entrepreneurs' innovation decisions both when investors are uncertainty-neutral SEU agents, and when they are MEU uncertainty-averse agents. By proceeding backward, we first determine the choice of the innovation intensity by the entrepreneurs, y_τ, and then solve for the initial choice of initiating the innovation process by incurring the discovery cost k_τ.

The implementation of the second stage of the innovation process requires entrepreneurs to raise capital from investors by selling equity in the capital markets at $t = 1$. Therefore, the choice of innovation intensity y_τ by a type-τ entrepreneur depends on her anticipation of the price that outside investors are willing to pay for their firms, that is on the market value of equity. This, in turn, depends on the beliefs held by investors on the success probability of the innovation, $p_\tau(\theta)$.

Lemma 3 *Entrepreneurs' firms are priced at their expected value, given investors' equilibrium beliefs, that is $V_\tau = p_\tau(\theta^a)y_\tau$ and $V_\tau = p_\tau(\theta^e)\,y_\tau$, $\tau \in \{A, B\}$, for uncertainty-averse and uncertainty-neutral investors, respectively. In equilibrium, it is (weakly) optimal for investors to hold the same portfolio: $\omega_A = \omega_B = 1$.*

Lemma 3 shows that, given our assumption of risk-neutrality, investors price equity at its expected value, given their beliefs. Investors' beliefs, however, depend on their attitude toward uncertainty, that is whether they are uncertainty-neutral investors or uncertainty-averse investors. Endogeneity of beliefs is critical because it will lead to different market valuation of equity, and thus, different behavior by entrepreneurs.

3.1 The Uncertainty-Neutral Case

As a benchmark, we start with the simpler case in which investors are uncertainty-neutral. When investors are uncertainty-neutral, equity prices depend only on their prior $\theta^e = \frac{1}{2}(\theta_0 + \theta_1)$ and on the level of innovation intensity, y_τ, chosen by the firm, giving

$$V_\tau = e^{\frac{1}{2}(\theta_0 - \theta_1)} y_\tau, \quad \text{for} \quad \tau \in \{A, B\}. \tag{6}$$

Eq (6) shows that equity value for an innovation of a type-τ does not depend on the innovation intensity decision of the other firm. This means that, under uncertainty-neutrality, there are no interactions between the choice of the innovation intensities by the two entrepreneurs. In this case, if the first stage of the project-idea was successful, entrepreneur τ's chooses the level of innovation intensity for the second stage, y_τ, by maximizing

$$\max_{y_\tau} \quad U_\tau^S \equiv V_\tau - c_\tau(y_\tau) = e^{\frac{1}{2}(\theta_0 - \theta_1)} y_\tau - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}. \tag{7}$$

From (7) it can immediately obtained that the optimal innovation intensity chosen by entrepreneur τ, y_τ, is equal to $y_\tau^* = \left[e^{\frac{1}{2}(\theta_0 - \theta_1)} Z_\tau\right]^{\frac{1}{\gamma}}$.[17] By direct substitution of y_τ^* into (7), we obtain that the ex-ante expected payoff for entrepreneur τ from initiating the innovation

[17] It is easy to veryfy that $V_\tau'' = -\frac{\gamma}{Z_\tau} y_\tau^{\gamma - 1} < 0$, so that first-order conditions are sufficient for a maximum.

process, and thus incurring into discovery discover k_τ, is equal to

$$EU_\tau^S = q_\tau \frac{\gamma}{1+\gamma} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} - k_\tau.$$

Thus, entrepreneur τ innovates at $t=0$ if $EU_\tau^S \geq 0$, leading to the following theorem.

Theorem 2 *When investors are uncertainty-neutral, enterpreneurs of type τ innovate iff*

$$k_\tau \leq k_\tau^S \quad \equiv q_\tau \frac{\gamma}{1+\gamma} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}, \quad \tau \in \{A, B\}.$$

Thus, innovation processes are independent.

Theorem 2 shows that when investors are uncertainty neutral, the investment decisions by the two entrepreneurs are effectively independent from each other, with no spillover effects. When investors are uncertainty averse, however, the innovation processes of the two firms are interconnected, as it will be showed below.

3.2 Uncertainty and Innovation

We now derive the optimal innovation decisions when investors are uncertainty averse. In this case, from (5), we know that the beliefs held by investors on the success probability of the second stage, $p_\tau(\theta^a)$, depend on the overall risk exposure of their portfolios. Specifically, beliefs held by uncertainty-averse investors are endogenous, and depend on the innovation intensities chosen by both firms, y_τ, and on their relative portfolio investment in the two firms, ω_A/ω_B. However, from Lemma 3, ambiguity-averse investors choose a balanced portfolio with $\omega_A = \omega_B$, which means that, in equilibrium, the market value of equity of each firms depends only on the level of innovation intensity chosen by both forms, y_τ, as follows.

Lemma 4 *If investors are uncertainty averse, the market value of entrepreneur τ's firm is*

$$V_\tau(\Pi) = \begin{cases} e^{\hat{\theta}_1 - \theta_1} y_\tau & y_\tau \leq e^{2\left(\theta^e - \hat{\theta}_1\right)} y_{\tau'} \\ e^{\frac{1}{2}(\theta_0 - \theta_1)} y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} & y_\tau \in \left(e^{2\left(\theta^e - \hat{\theta}_1\right)} y_{\tau'}, e^{2\left(\theta^e - \hat{\theta}_0\right)} y_{\tau'}\right) \\ e^{\hat{\theta}_0 - \theta_1} y_\tau & y_\tau \geq e^{2\left(\theta^e - \hat{\theta}_0\right)} y_{\tau'} \end{cases}, \tag{8}$$

where y_τ is the innovation intensity selected by entrepreneur of type-τ, with $\tau, \tau' \in \{A, B\}$, $\tau \neq \tau'$.

Lemma 4 shows that the market value of equity of one firm depends on the level of innovation intensity chosen by its entrepreneur as well as on the level chosen by the other firm. The linkage between the market value of the two firms occurs through investors' beliefs. In particular, from (5) an increase of firm-τ innovation intensity, y_τ, will increase the relative exposure of investors to firm-τ risk relative to firm-τ' risk, making (all else equal) investors relatively more pessimistic about firm-τ success probability and, correspondingly, relatively more optimistic about about firm-τ' success probability. This implies that firm value is a (weakly) increasing function of the innovation intensities of both firms. Further, if one of the two firms does not innovate, which means that the level of innovation intensity for that firm is necessarily equal to zero, the market value of equity of the other firm will be determined at the worst case scenario for that firm, that is $V_\tau(\Pi) = \min_\theta p_\tau(\theta) y_\tau$. This interaction between the market values of the equity of the two firms, caused by investors' beliefs, creates a strategic externality between the two entrepreneurs, which will be critical in the analysis below.

We can now determine the optimal level of innovation intensity for entrepreneur τ. If the first stage of the project-idea was successful, entrepreneur τ's chooses the level of innovation intensity for the second stage, y_τ, by maximizing

$$\max_{y\tau} U_\tau^M \quad \equiv V_\tau(\Pi) - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}, \tag{9}$$

where $\Pi = \{y_A, y_B, w_0 - V_A - V_B\}$, since $\omega_A = \omega_B = 1$. To simplify the exposition, in what follows we assume Z_A and Z_B do not a value too dissimilar from each other, so that the investor has interior beliefs in equilibrium. Formally, we assume that $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi \equiv \frac{1}{4} e^{2\left(\theta^e - \hat{\theta}_0\right)(\gamma+1)} \left(1 + \frac{1}{2\gamma}\right)^{2\gamma}$ so that, if both firms have successful first-stage projects they find it optimal to produce output levels that result in interior beliefs for the investors.

The solution to problem (9) depends on whether one or both firms decide to initiate the innovation process and pay the discovery costs k_τ and, if they do so, whether they are successful at the first stage of the innovation process. There are four states of the world that we need to analyze: (i) when both entrepreneurs had a successful first stage, state SS; (ii) when only one entrepreneur has a successful first-stage, state SF with the symmetric FS state, (iii) when both entrepreneur fail in the first stage and no innovation can take place, state FF. Since the last state FF is trivial, we now focus on the other three.

3.2.1 Only One Firm Has Successful First-Stage Project, State SF

Consider first the case in which only entrepreneur of type-τ had a successful first-stage project-idea, state SF. For future reference, note that this state may emerge either because the other entrepreneur of type-τ', with $\tau' \neq \tau$, has not initiated the innovation process (that is, she did not sustain the discovery cost k_τ), or because the first stage of the innovation process has been unsuccessful, if the entrepreneur has initiated it.

Lemma 5 *If only entrepreneur of type-τ has a successful first stage project-idea (state*

SF), she selects innovation intensity equal to

$$y_\tau^{M,SF} = \left[e^{\hat{\theta}_0-\theta_1} Z_\tau\right]^{\frac{1}{\gamma}}; \tag{10}$$

the market value of the entrepreneur's firm is equal to

$$V_\tau^{M,SF} = e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}, \tag{11}$$

giving a continuation utility for the entrepreneur equal to

$$U_\tau^{M,SF} \equiv e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}. \tag{12}$$

If only one entrepreneur successfully develops a first-stage project, there will only be one type of uncertain innovation available to investors. In this case, from (5) investors will believe the worst-case scenario about that innovation type, resulting in pessimistic beliefs and low equity valuations. Therefore, the entrepreneur will chose a low level of innovation intensity, consistent with the endogenously pessimistic beliefs held by investors.

3.2.2 Both Firms Have Successful First-Stage Projects, State SS

If both entrepreneurs have successful first-stage projects, market valuation is given in Lemma 4, which leads to the following theorem.

Lemma 6 *Let* $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$. *If both entrepreneurs innovate and have a successful first stage (state SS), they select innovation intensities equal to*

$$y_\tau^{M,SS}(y_{\tau'}) = \left[\frac{Z_\tau}{2} e^{\frac{1}{2}(\theta_0-\theta_1)} (y_{\tau'})^{1/2}\right]^{\frac{1}{\gamma+\frac{1}{2}}}, \quad \text{with} \quad \tau \neq \tau', \text{ and } \tau, \tau' \in \{A, B\}. \tag{13}$$

Lemma 6 shows that there is strategic complementarity in entrepreneurs' production decisions. In particular, it is easy to see from (13) that an entrepreneur's choice of innovation intensity, $y_\tau^{M,SS}(y_{\tau'})$, is an increasing function of the other entrepreneur's choice of, $y_{\tau'}$. The strategic complementarity originates in investors' uncertainty aversion because, from Lemma 2, they treat different innovations as complements. This complementarity in beliefs is then transferred from investors' preferences to entrepreneurs' innovation decisions.

We can now determine the equilibrium levels of innovation intensities chosen by the two entrepreneurs in the SS state.

Theorem 3 *If both entrepreneurs innovate and have successful first-stage projects, state SS, the Nash-Equilibrium level of innovation intensities for an entrepreneur of type τ, with $\tau \in \{A, B\}$, is equal to:*

$$y_\tau^{M,SS} = \left[\frac{1}{2} e^{\frac{1}{2}(\theta_0-\theta_1)} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}. \tag{14}$$

In addition, in equilibrium, firm value for both firms is equal to

$$V_\tau^{M,SS} = 2^{-\frac{1}{\gamma}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} \left(Z_\tau Z_{\tau'}\right)^{\frac{1}{2\gamma}}, \tag{15}$$

and continuation utility is equal to

$$U_\tau^{M,SS} = 2^{-\frac{1}{\gamma}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} \left(Z_\tau Z_{\tau'}\right)^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}. \tag{16}$$

The following corollary compares the equilibrium values when one or both entrepreneurs have successful first-stage projects.

Corollary 1 *Entrepreneurs are better off when both entrepreneurs have successful first-stage projects: $U_\tau^{M,SS} > U_\tau^{M,SF}$. If entrepreneurs productivities are not too far apart,*

$\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi_1}, \psi_1\right)$, *equity values are higher when both entrepreneurs have successful first-stage projects:* $V_\tau^{M,SS} > V_\tau^{M,SF}$. *If entrepreneurs productivities are sufficiently close togerther,* $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi_2}, \psi_2\right)$, *entrepreneurs innovate with greater intensity when both have successful first-stage projects:* $y_\tau^{M,SS} > y_\tau^{M,SF}$. *Finally,* $\psi_2 < \psi_1 < \psi$.

An important implication of Corollary 1 is that, if entrepreneurs' producitivites are close enough together, because of the complementarity of beliefs due to uncertainty aversion, investors value innovations of one type of innovation when they can invest also in the other type of innovation, yielding $V_\tau^{*M,SS} > V_\tau^{*M,SF}$.

3.3 The Innovation Decision

In the previous sections we have shown that investors' uncertainty aversion affects equity valuations and generates strategic complementarity in the interim choice of innovation intensity, y_τ. Proceeding backward, the interim strategic complementarity of the choice of innovation intensity translates into a strategic complementarity in the entrepreneurs' decisions to innovate at the beginning of the period, that is to incur in the discovery cost k_τ. The expected utility for an entrepreneur of type-τ from sustaining at $t = 0$ the initial discover cost k_τ and, thus, initiating the innovation process is equal to

$$
\begin{aligned}
EU_\tau^M &= (q_\tau q_{\tau'} + r)U_\tau^{M,SS} + (q_\tau(1 - q_{\tau'}) - r)U_\tau^{M,SF} - k_\tau \\
&= (q_\tau q_{\tau'} + r)2^{-\frac{1}{\gamma}} e^{\frac{1}{2}(\theta_0 - \theta_1)\frac{1+\gamma}{\gamma}} (Z_\tau Z_{\tau'})^{\frac{1}{2\gamma}} \frac{2\gamma + 1}{2\gamma + 2} + \\
&\quad (q_\tau(1 - q_{\tau'}) - r)e^{\left(\hat{\theta}_0 - \theta_1\right)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1 + \gamma} - k_\tau,
\end{aligned}
$$

for $\tau, \tau' \in \{A, B\}$ and $\tau \neq \tau'$. We now characterize the Nash Equilibrium of the innovation decision at the beginning of the period, $t = 0$.

Theorem 4 *There are threshold levels* $\{\underline{k}_\tau, \bar{k}_\tau\}_{\tau\in\{A,B\}}$ *(defined in the appendix) with* $\underline{k}_\tau < \bar{k}_\tau$*, such that: (i) iff* $k_\tau \leq \underline{k}_\tau$*, entrepreneur of type* τ *always innovates; (ii) if* $k_\tau \geq \bar{k}_\tau$*, entrepreneur of type* τ *never innovates; (iii) If* $k_\tau \in (\underline{k}_\tau, \bar{k}_\tau)$ *entrepreneur of type* τ *innovates if* $k_{\tau'} \leq \underline{k}_{\tau'}$*, and she does not innovate if* $k_{\tau'} \geq \bar{k}_{\tau'}$*; (iv) if* $k_\tau \in (\underline{k}_\tau, \bar{k}_\tau)$ *for both* $\tau \in \{A, B\}$*, there are multiple equilibria, one where both entrepreneurs innovate and one where neither innovate. The equilibrium where both entrepreneurs innovate dominates the equilibrium where none of the entrepreneurs innovate.*

For very small levels of discovery costs, $k_\tau \leq \underline{k}_\tau$, it is a dominant strategy for entrepreneur τ to innovate. For very large levels of discovery costs, $k_\tau \geq \bar{k}_\tau$, it is a dominant strategy for entrepreneur τ to not innovate. For intermediate levels of discovery costs, $k_\tau \in (\underline{k}_\tau, \bar{k}_\tau)$, entrepreneur τ wishes to innovate only if the other entrepreneur innovates as well. Theorem 4 shows the strategic complementarity in entrepreneurs' innovation decisions.

When both entrepreneurs have intermediate levels of the discovery cost, there are multiple equilibria, with and without innovation. In this case, entrepreneurs face a classic "assurance game," in which there is a Pareto dominant equilibrium, where both entrepreneurs innovate, yet there is also an inefficient, Pareto-inferior equilibrium where neither entrepreneurs innovate. Multiplicity of equilibria depends on the fact that it is profitable for one entrepreneur to innovate only if he expects the other entrepreneur to innovate as well. Such multiplicity of equilibria in the innovation game is the direct outcome of investors' uncertainty aversion.

We conclude this section by characterizing the impact of the model's parameters on the threshold leves $\{\underline{k}_\tau, \bar{k}_\tau\}_{\tau\in\{A,B\}}$.

Corollary 2 *The threshold levels* $\{\bar{k}_\tau\}_{\tau\in\{A,B\}}$ *are increasing functions of* $q_\tau, q_{\tau'}, Z_\tau, Z_{\tau'}$ *and* r*, and the threshold levels* $\{\underline{k}_\tau\}_{\tau\in\{A,B\}}$ *are increasing functions of* q_τ *and* Z_τ*.*

Corollary (2) has the interesting implication that an increase in one firm's probability of success, q_τ, makes not only that firm, but also other firms, more willing to attempt first-stage discovery of a product-idea. This follows because the strategic complementarity induced by uncertainty aversion. In the absence of uncertainty aversion, an increase in the probability of discovery affects only that entrepreneur, with no effect on other entrepreneurs. Corollary (2) also shows that entrepreneurs are more willing to innovate if her innovation is more related to other entrepreneurs' innovations, that is r is greater. This happens because greater degree of relatedness increases the probability that both project-ideas are simultanously successful in the first-stage, increasing the market value of the innovations. Finally, Corollary 2 also shows that an increase in productivity of an entrepreneur increases not only that entrepreneur's willingness to innovate, but also makes other entrepreneurs willing to innovate as well.

4 Empirical Implications

Our paper has several novel empirical implications.

1. Innovation waves. The strategic complementarity between entrepreneurs' innovation decisions creates in our model the possibility of innovation waves. An innovation wave occurs if an entrepreneur's decision to initiate the innovation process, and thus to undertake the first stage of her project-idea, has the effect of inducing also the other entrepreneur to do the same. This can happen, for example, when a positive shock in the project idea of one entrepreneur lowers the discovery cost from $k_\tau > \hat{k}_\tau$ to $k_\tau < \bar{k}$, while $k_{\tau'} \in \left(\bar{k}_{\tau'}, \hat{k}_{\tau'}\right)$, $\tau \neq \tau'$. In this case, if initially $k_\tau > \hat{k}$, it is not profitable for the entrepreneur of type τ to initiate the innovation process, which will discourage the entrepreneur of type τ' to do the same. If now the discover costs of entrepreneur of type τ are lower to $k_\tau < \bar{k}$, it becomes optimal for the entrepreneur to initiate the innovate process. The decision of entrepreneur τ to initiate the innovation process makes it now profitable for entrepreneur of type τ' to initiate the innovation process as well, in anticipation of the possibility of higher equity prices if both entrepreneurs are successful. Thus, a positive idiosyncratic shock to the technology of an entrepreneur spills over to the other entrepreneur, triggering an innovation wave. Note that the beneficial spillover effect is more likely to occur the greater the degree of relatedness of the two technologies, that is the greater the value of the parameter r.

2. Innovation waves, investors' optimism, and hot IPO markets. In our model, the market value of an entrepreneur's firm is greater when there are two firms in the market, rather than only one. This is because uncertainty-averse investors are more optimistic when they can invest in the equity of both firms, rather than in one firm only, leading to higher equity valuations. Given the discussion on point 1 above, this means that innovation waves will be associated with strong investors' sentiment toward innovations and, thus, booms in the equity markets of technology firms. This means that innovation waves are associated with hot IPO markets. In additions, innovation waves and hot IPO markets are more likely to occur in related industries.

3. Innovation waves and venture capitalists. An additional implication of our model is a new role for venture capitalists. In the case in which discovery costs fall in the intermediate range, $k_\tau \in \left(\bar{k}_\tau, \hat{k}_\tau\right)$, entrepreneurs face an "assurance game" in that each entrepreneur will be willing to incur the discovery cost and thus innovate only if she is assured that also the other entrepreneur will do the same. Lacking such assurance, entrepreneurs may be confined to the inefficient equilibrium with no innovation. In this setting, a venture capitalist may indeed play the positive role to address the coordination failure among entrepreneurs. By investing in both firms, the venture capitalist can help coordination among entrepreneurs and lead to greater innovation. In addition, as discussed above, coordination among entrepreneurs' innovation activities will be associated with greater equity market valuations. These observations imply that venture capital activity will be associated with innovation waves and greater equity valuations.

5 Conclusion

In this paper, we show that uncertainty aversion generates innovation waves. Uncertainty aversion causes investors to treat different uncertain lotteries as complements, a property that we refer to as uncertainty hedging. Uncertainty hedging by investors produces strategic complementarity in entrepreneurial behavior, producing innovation waves. Specifically, when one entrepreneur has a successful first-stage project, equity valuation, entrepreneur utility, and the intensity of innovation increase for other entrepreneurs as well. Thus, entrepreneurs are more willing to innovate if they expect other entrepreneurs are going to innovate as well, resulting in multiple equilibria. Our model can thus explain why there are some periods when investment in innovation is "hot," and venture capitalists are more willing to invest in risky investment projects tainted by significant uncertainty.

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A Appendix

Proof of Theorem 1. Let $V_\mu = qE_\mu[u(y_1)] + (1-q)E_\mu[u(y_1)]$, and define $\mu_1 = \arg\min E_\mu[u(y_1)]$, $\mu_2 = \arg\min E_\mu[u(y_2)]$, and $\mu_q = \arg\min V_\mu$. Thus, $E_{\mu_1}[u(y_1)] \leq E_{\mu_q}[u(y_1)]$ and $E_{\mu_2}[u(y_2)] \leq E_{\mu_q}[u(y_2)]$, so $qE_{\mu_1}[u(y_1)] + (1-q)E_{\mu_2}[u(y_2)] \leq qE_{\mu_q}[u(y_1)] + (1-q)E_{\mu_q}[u(y_2)] = \min V_\mu$. Thus, (3) holds. Because uncertainty-neutral agents can be modeled as uncertainty-averse agents with a singleton for their core of beliefs, the inequality holds with equality in the absence of uncertainty aversion. ∎

Proof of Lemma 1. Define $u(\theta;\Pi) = e^{\theta-\theta_1}\omega_A y_A + e^{\theta_0-\theta}\omega_A y_B + w_0 - \omega_A V_A - \omega_B V_B$, so that $U(\Pi) = \min_{\theta\in C}\{u(\theta;\Pi)\}$. Thus, $u_\theta = e^{\theta-\theta_1}\omega_A y_A - e^{\theta_0-\theta}\omega_B y_B$, and $u_{\theta\theta} = e^{\theta-\theta_1}\omega_A y_A + e^{\theta_0-\theta}\omega_B y_B$. Because $u_{\theta\theta} > 0$, u is convex in θ, so first order conditions are sufficient for a minimum. $u_\theta = 0$ iff $\theta = \tilde{\theta}^a$ where

$$\tilde{\theta}^a(\Pi) = \frac{1}{2}(\theta_0+\theta_1) + \frac{1}{2}\ln\frac{\omega_B y_B}{\omega_A y_A}.$$

Thus, if $\tilde{\theta}^a(\Pi) \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$, $\theta^a = \tilde{\theta}^a$ (because $\tilde{\theta}^a$ minimizes u). If $\tilde{\theta}^a < \hat{\theta}_0$, $u_\theta > 0$ for all $\theta \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$, so $\theta^a = \hat{\theta}_0$. Similarly, if $\tilde{\theta}^a > \hat{\theta}_1$, $u_\theta < 0$ for all $\theta \in \left[\hat{\theta}_0, \hat{\theta}_1\right]$, so $\theta^a = \hat{\theta}_1$. Therefore, (5) is the worst-case scenario for the investor. ∎

Proof of Lemma 3. Each investor's objective function is $U(\Pi) = \min_{\theta\in C} u(\theta;\Pi)$ where $u(\theta;\Pi) = e^{\theta-\theta_1}\omega_A y_A + e^{\theta_0-\theta}\omega_B y_B + w_0 - \omega_A V_A - \omega_B V_B$. Thus, for $\tau \in \{A, B\}$,

$$\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial\omega_\tau} + \frac{\partial u}{\partial\theta}\frac{d\theta}{d\omega_\tau}$$

If investors are uncertainty-neutral, they believe $C = \{\theta^e\}$, so the second term disappears ($\theta = \theta^e$, so it is constant). If investors are uncertainty averse, θ^a solves the minimization problem, so either $\frac{\partial u}{\partial\theta} = 0$ (an interior solution) or $\frac{d\theta}{d\omega_A} = 0$ (a corner solution). Thus, $\frac{\partial u}{\partial\theta}\frac{d\theta}{d\omega_\tau} = 0$, so that $\frac{dU}{d\omega_\tau} = \frac{\partial u}{\partial\omega_\tau}$ for $\tau \in \{A, B\}$.

$$\frac{\partial u}{dy_A} = e^{\theta^a-\theta_1}y_A - V_A$$

and

$$\frac{du}{dy_B} = e^{\theta_0-\theta^a}y_B - V_B$$

Thus, market clearing requires that $V_A = e^{\theta^a-\theta_1}y_A$ and $V_B = e^{\theta_0-\theta^a}y_B$. Because $p_A(\theta^a) = e^{\theta^a(\Pi)-\theta_1}$ and $p_B(\theta^a) = e^{\theta_0-\theta^a(\Pi)}$, it follows that $V_A = p_A(\theta^a)y_A$ and $V_B = p_B(\theta^a)y_B$ (The proof is identical for SEU, with θ^e instead of θ^a). Note that it is WLOG optimal for all investors to set $\omega_A = \omega_B = 1$, because innovations are priced at expected value given market beliefs. Further, if investors are uncertainty-averse, they will hold identical positions in the risky portfolio (formally, $\frac{\omega_A}{\omega_B}$ is constant across all investors), because there would be gains from trade if they did not. ∎

Proof of Lemma 4. Solve the problem in three cases: when $\theta^a(\Pi) = \hat{\theta}_0$, when $\theta^a(\Pi) = \hat{\theta}_1$, and when $\theta^a(\Pi) \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$.

From Lemma 1, $\theta^a(\Pi) = \hat{\theta}_0$ iff $\tilde{\theta}^a(\Pi) \leq \hat{\theta}_0$ iff $y_A \geq e^{2\left(\theta^e-\hat{\theta}_0\right)}y_B$. Thus, if $y_A \geq e^{2\left(\theta^e-\hat{\theta}_0\right)}y_B$, $V_A = p_A\left(\hat{\theta}_0\right)y_A$ and $V_B = p_B\left(\hat{\theta}_0\right)y_B$. Similarly, $\theta^a(\Pi) = \hat{\theta}_1$ iff $\tilde{\theta}^a(\Pi) \geq \hat{\theta}_1$ iff $y_A \leq e^{2\left(\theta^e-\hat{\theta}_1\right)}y_B$. Thus, if $y_A \leq e^{2\left(\theta^e-\hat{\theta}_1\right)}y_B$, $V_A = p_A\left(\hat{\theta}_1\right)y_A$ and $V_B = p_B\left(\hat{\theta}_1\right)y_B$. Finally, from Lemma 1, $\theta^a(\Pi) \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$ iff

$\tilde{\theta}^a(\Pi) \in \left(\hat{\theta}_0, \hat{\theta}_1\right)$ iff $y_A \in \left(e^{2\left(\theta^e-\hat{\theta}_1\right)}y_B, e^{2\left(\theta^e-\hat{\theta}_0\right)}y_B\right)$. Because $\theta^a(\Pi) = \tilde{\theta}^a(\Pi)$ on this region, $p_\tau(\theta^a(\Pi)) = e^{\frac{1}{2}(\theta_0-\theta_1)}y_{\tau'}^{\frac{1}{2}}y_\tau^{-\frac{1}{2}}$,which implies that the market values entrepreneur τ's firm at $V_\tau = e^{\frac{1}{2}(\theta_0-\theta_1)}y_{\tau'}^{\frac{1}{2}}y_\tau^{\frac{1}{2}}$. The piecewise function immediately follows because p_A is increasing in θ but p_B is decreasing in θ. There is strategic complementarity in production because $\frac{\partial V_\tau}{\partial y_{\tau'}} \geq 0$ for $\tau' \neq \tau$, with strict inequality for $y_A \in \left(e^{2\left(\theta^e-\hat{\theta}_1\right)}y_B, e^{2\left(\theta^e-\hat{\theta}_0\right)}y_B\right)$. ■

Proof of Lemma 5. Suppose that only entrepreneur A has a successful first-stage project-idea (the case with entrepreneur B follows symmetrically), so $y_B = 0$. By Lemma 1, $\tilde{\theta}^a = -\infty$, so $\theta^a = \hat{\theta}_0$. By Lemma 3, $p_A(\theta_0) = e^{\hat{\theta}_0-\theta_1}$, so $V_A = e^{\hat{\theta}_0-\theta_1}y_A$. Thus, entrepreneur A's payoff is

$$U_A = e^{\hat{\theta}_0-\theta_1}y_A - \frac{1}{Z_A(1+\gamma)}y_A^{1+\gamma} - k_A$$

Note that $\frac{\partial U_A}{\partial y_A} = e^{\hat{\theta}_0-\theta_1} - \frac{1}{Z_A}y_A^\gamma$, and $\frac{\partial^2 U_A}{\partial y_A^2} = -\frac{\gamma}{Z_A}y_A^{\gamma-1} < 0$, so FOCs are sufficient for a maximum. Thus, entrepreneur A selects $y_A^{M,SF} = \left[e^{\hat{\theta}_0-\theta_1}Z_A\right]^{\frac{1}{\gamma}}$, sells for $V_A^{M,SF} = e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}}Z_A^{\frac{1}{\gamma}}$, and earns continuation payoff $U_A^{M,SF} = e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}}Z_A^{\frac{1}{\gamma}}\frac{\gamma}{1+\gamma}$. ■

Proof of Lemma 6. Suppose it is optimal for entrepreneurs to produce output resulting in interior beliefs: $y_A \in \left(e^{\theta_0-\theta_1}y_B, e^{\theta_1-\theta_0}y_B\right)$, which will be optimal because the assumptions on Z_A and Z_B. For $\tau \in \{A, B\}$ and $\tau' \neq \tau$, when entrepreneur τ' produces y_τ, entrepreneur τ produces y_τ and earns continuation utility

$$U_\tau = e^{\frac{1}{2}(\theta_0-\theta_1)}y_\tau^{\frac{1}{2}}y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau(1+\gamma)}y_\tau^{1+\gamma}.$$

Thus, $\frac{\partial U_\tau}{\partial y_\tau} = \frac{1}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}y_\tau^{-\frac{1}{2}}y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau}y_\tau^\gamma$ and $\frac{\partial^2 U_\tau}{\partial y_\tau^2} = -\frac{1}{4}e^{\frac{1}{2}(\theta_0-\theta_1)}y_\tau^{-\frac{3}{2}}y_\tau^{\frac{1}{2}} - \frac{\gamma}{Z_\tau}y_\tau^{\gamma-1}$. Because $\frac{\partial^2 U_\tau}{\partial y_\tau^2} < 0$, FOCs are sufficient for a local maximum. Thus, $y_\tau = \left[\frac{Z_\tau}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$.

On this region, optimal output by one firm is strictly increasing in the output of the other firm. Inspection of the revenue function from Lemma 4 immediately shows that entrepreneur A's problem is locally concave almost everywhere, the exception being at $y_A = e^{\theta_1-\hat{\theta}_0}y_B$. Because there is a kink at $y_A = e^{\theta_1-\hat{\theta}_0}y_B$, there may be multiple critical points, resulting in a discontinuous best response function. Thus, there are some parameter values for which there is no pure strategy equilibrium. However, it can be verified (after messy calculations) that so long as $\frac{Z_A}{Z_B} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi = \frac{1}{4}e^{2\left(\theta^e-\hat{\theta}_0\right)(\gamma+1)}\left(1+\frac{1}{2\gamma}\right)^{2\gamma}$, if both firms enter, there is a unique equilibrium – it is optimal for the firms to produce output levels such that investors have interior beliefs. Thus, on this region, entrepreneurs' best response functions satisfy $y_\tau^{M,SS}(y_{\tau'}) = \left[\frac{Z_A}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$. ■

Proof of Theorem 3. In equilibrium, the two entrepreneurs select innovation intensity optimally, given the intensity the other entrepreneur is innovating. From Lemma 6, the best response functions are $y_\tau^{M,SS}(y_{\tau'}) = \left[\frac{Z_A}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}y_{\tau'}^{\frac{1}{2}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}$. Because entrepreneur τ' also selects intensity optimally, selecting $y_{\tau'}^{M,SS}(y_\tau)$, it follows that

$$y_\tau = \left[\frac{Z_\tau}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}\left[\frac{Z_{\tau'}}{2}e^{\frac{1}{2}(\theta_0-\theta_1)}y_\tau^{\frac{1}{2}}\right]^{\frac{1}{2\gamma+1}}\right]^{\frac{1}{\gamma+\frac{1}{2}}}.$$

After some messy calculations, this holds iff

$$y_\tau^{M,SS} = \left[\frac{1}{2} e^{\frac{1}{2}(\theta_0-\theta_1)} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$$

for $\tau \in \{A, B\}$ and $\tau' \neq \tau$.

Because the market price is $V_\tau^{M,SS} = e^{\frac{1}{2}(\theta_0-\theta_1)} y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}}$, it follows that

$$V_\tau^{M,SS} = 2^{-\frac{1}{\gamma}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{\gamma+1}{\gamma}} \left[Z_\tau Z_{\tau'}\right]^{\frac{1}{2\gamma}}.$$

Similarly, entreprenuer τ earns continuation utility $U_\tau^{M,SS} = e^{\frac{1}{2}(\theta_0-\theta_1)} y_\tau^{\frac{1}{2}} y_{\tau'}^{\frac{1}{2}} - \frac{1}{Z_\tau(1+\gamma)} y_\tau^{1+\gamma}$, which can be expressed as

$$U_\tau^{M,SS} = \frac{1}{2^{\frac{1}{\gamma}}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2},$$

for $\tau \in \{A, B\}$ and $\tau' \neq \tau$. Thus, there are strategic complementarities in production and profit of the firms. ■

Proof of Corollary 1. Recall $U_\tau^{M,SS} = \frac{1}{2^{\frac{1}{\gamma}}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{2\gamma}} Z_{\tau'}^{\frac{1}{2\gamma}} \frac{2\gamma+1}{2\gamma+2}$ and $U_\tau^{M,SF} = e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}} \frac{\gamma}{1+\gamma}$. Thus, $U_\tau^{M,SS} > U_\tau^{M,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > \frac{1}{\psi}$ where $\psi = \frac{1}{4} e^{2\left(\theta^e-\hat{\theta}_0\right)(\gamma+1)} \left(1+\frac{1}{2\gamma}\right)^{2\gamma}$. Recall that we assumed $\frac{Z_{\tau'}}{Z_\tau} \in \left(\frac{1}{\psi}, \psi\right)$ where $\psi = \frac{1}{4} e^{2\left(\theta^e-\hat{\theta}_0\right)(\gamma+1)} \left(1+\frac{1}{2\gamma}\right)^{2\gamma}$, so this is always satisfied – entrepreneurs are better off when other entrepreneurs have a successful first-stage project.

Recall that $V_\tau^{M,SS} = 2^{-\frac{1}{\gamma}} e^{\frac{1}{2}(\theta_0-\theta_1)\frac{\gamma+1}{\gamma}} \left[Z_\tau Z_{\tau'}\right]^{\frac{1}{2\gamma}}$ and $V_\tau^{M,SF} = e^{\left(\hat{\theta}_0-\theta_1\right)\frac{1+\gamma}{\gamma}} Z_\tau^{\frac{1}{\gamma}}$. After some messy algebra, it can be shown that $V_\tau^{M,SS} > V_\tau^{M,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4 e^{2\left(\hat{\theta}_0-\theta^e\right)(1+\gamma)}$. Define $\psi_1 = \frac{1}{4} e^{2\left(\theta^e-\hat{\theta}_0\right)(1+\gamma)}$.

Finally, $y_\tau^{M,SS} = \left[\frac{1}{2} e^{\frac{1}{2}(\theta_0-\theta_1)} Z_\tau^{\frac{2\gamma+1}{2\gamma+2}} Z_{\tau'}^{\frac{1}{2\gamma+2}}\right]^{\frac{1}{\gamma}}$ and $y_\tau^{M,SF} = \left[e^{\hat{\theta}_0-\theta_1} Z_\tau\right]^{\frac{1}{\gamma}}$. After some messy algebra, it can be shown that $y_\tau^{M,SS} > y_\tau^{M,SF}$ iff $\frac{Z_{\tau'}}{Z_\tau} > 4^{\gamma+1} e^{2\left(\hat{\theta}_0-\theta^e\right)(1+\gamma)}$. Thus, define $\psi_2 = \left[\frac{1}{4}\right]^{\gamma+1} e^{2\left(\theta^e-\hat{\theta}_0\right)(1+\gamma)}$. Further, because $\gamma > 0$, it follows immediately that $\psi_2 < \psi_1 < \psi$. ■

Proof of Theorem 4. If only entrepreneur τ innovates, he earns payoff $EU_\tau^M = q_\tau U_\tau^{M,SF} - k_\tau$ (Lemma 5). Thus, if an entrepreneur does not expect the other entreprenuer to innovate, he will innovate iff $k_\tau \leq \underline{k}_\tau \equiv q_\tau U_\tau^{M,SF}$. Conversely, if the other entrepreneur innovates, entrepreneur τ earns payoff $EU_\tau^M = (q_\tau q_{\tau'} + r) U_\tau^{M,SS} + \left[q_\tau(1-q_{\tau'}) - r\right] U_\tau^{M,SF} - k_\tau$ if he innovates as well. Thus, if the other entrepreneur innovates, entreprenuer τ will innovate iff $k_\tau \leq \bar{k}_\tau \equiv (q_\tau q_{\tau'} + r) U_\tau^{M,SS} + \left[q_\tau(1-q_{\tau'}) - r\right] U_\tau^{M,SF}$. By Corollary 1, $U_\tau^{M,SF} < U_\tau^{M,SS}$, so it follows that $\underline{k}_\tau < \bar{k}_\tau$ (because the coefficients on the terms in $\bar{k}_\tau$ sum to q_τ). ■

Proof of Corollary 2. Comparative Statics follow immediately from inspection of the expressions. ■